Final Report: Mathematical aspects of turbulence: where do we stand?

January 4th – June 27th 2022

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The planning period prior to this programme (TUR) coincided with the pandemic crisis and global lockdown. It was also sadly marked by the death in May 2021 of one our much-loved organizers, namely Charles Doering of the University of Michigan. The beginning of the programme in the first week of January 2022 also coincided with the peak of the Omicron variant. Thus, all the talks of our opening Workshop TURW01 (Jan 4th-7th) were held on-line, during which senior speakers were asked to present their grand challenges.

Early in the physical program, Mike Cullen (Met Office) set the stage in a series of 4 lectures on atmospheric dynamics. Simons Research Fellows, Anagha Madhu and Antoine Remond-Tiedrez (both Cambridge and INI), organized the Junior Isaac Newton Crossover Seminar (JINX), jointly with the parallel Kinetic Theory programme (FKT), while Simon Markfelder (Cambridge) gave a mini-course on Convex Integration. This a technique devised in the last decade to analyze rough solutions of the Euler equations. General programme highlights were the two superb Distinguished Lectures (the Kirk and Rothschild) delivered respectively by Rama Govindarajan (ICTS Bangalore) on *Mammatus Clouds and Sedimentation Instabilities* and by Rupert Klein (Free University of Berlin) on *How Mathematics Helps Structuring Climate Discussions*.

Mathematical Analysis Quarter: Analysis of the Euler and Navier-Stokes equations (and other models of fluid dynamics) contributes to our understanding of turbulence in many ways. Rigorous results on regularity often exclude physical singularities of particular types which can influence coarser-scale modelling. Progress on questions that at first appear abstract, such as the Onsager Conjecture (minimal smoothness for conservation of energy in the Euler equations), can provide insights towards foundations of the classical theory of turbulence, namely the dissipation anomaly (finite energy dissipation in the limit of zero viscosity). Moreover, analysis of boundary layers is central to many asymptotic approaches to

fluid flows, and key mathematical questions remain which are closely related to convergence of Navier-Stokes to Euler flows.

The centerpiece of this part of the program was the second workshop (TURW02) and the first held in hybrid mode. Representing a diverse group of researchers, approximately 100 participants attended TURW02, of which about 45 were physically present at the INI. The following topics were emphasized during the workshop: energy dissipation in solutions of the Euler equations, statistical solutions, inviscid limit and boundary layers, stability, ill-posedness and singularity formation for both the Euler and Navier-Stokes equations. The role of the pressure and rigorous notions of intermittency were also emphasized. Speakers were asked to make their presentation accessible to a broad audience of both pure and applied mathematicians. Presentations by both leaders in the field and more junior researchers from the Americas, Asia, and Europe were followed each day by a period of discussion led by the organizers and held in hybrid mode.

The workshop was part of a comprehensive plan for the long-term programme of integrating rigorous analysis of turbulent flows with modelling, experiments and simulations. In this spirit, the workshop was complemented with regular seminars, scheduled almost on a weekly basis, open to all program participants, whether physical or remote. There was also a fruitful interaction with DAMTP and DPMMS; programme participants attended and spoke in the PDE Seminar. Particular attention was made to maintain a balance of diversity, in terms of gender, geographic provenance and career stage. For instance, 5 out of the 20 speakers for Workshop 2 were women and 2 were early career researchers, with a similar mix for the weekly seminars.

Turbulent transport, mixing and scaling processes quarter: Transport and mixing are among the most important aspects of turbulent fluid flows. The movement of mechanical and thermal energy and material underlies the most basic dynamical phenomena in astrophysics, geophysics and the environment, as well as in biology and myriad applications in engineering. Therefore, it is important to understand both the fundamental physics underlying the dynamics of these processes and novel mathematical approaches for their analysis.

The speakers and participants of Workshop TURW03 (7th-11th March) included leading mathematicians, who aimed to prove rigorous results in this challenging

area, together with physicists and engineers, who study turbulent flows, computationally, theoretically and experimentally. The main focus was on quantitative modelling and analysis of turbulent transport and mixing. Topics that were discussed included recent advances in irregular transport and in geometric analysis, developments and novel applications from optimization, control and physics. Climate applications, including the use of machine learning and methods in artificial intelligence, were also discussed.

Wall-bounded flows quarter: Understanding the dynamical processes involved in initiating and maintaining wall-confined turbulent flows remains a considerable practical challenge. Technologically, the most pressing concern is reducing frictional turbulent drag on walls (overcoming this is estimated to represent 5% of mankind's energy expenditure). While the multi-scaled structure of turbulent flows is well-documented near a wall (e.g. the `law of the wall' goes back to the 1930s), there is still no predictive *a priori* theory for it.

Workshop TURW04 (28th March-1st April) brought together over 130 applied mathematicians, physicists and engineers and contained 42 invited 30-minute presentations. A number of themes ran through the meeting. The first was the continuing progress in treating transitional and weakly turbulent flows as a huge dynamical system. This was a nascent 'hot' topic in the previous (2008) INI meeting and recent developments, including treating larger flow domains and spatiallyevolving boundary layer flows, were discussed. A number of talks discussed advances made in characterizing complex boundary conditions (and their effect on turbulence) in flows that either originate from wall roughness in engineering applications, or from the atmospheric surface layer. Another theme was the study of polymer-enhanced flows, long known to reduce frictional wall drag even at very low polymer concentrations but increasingly studied for the varied turbulent states that can arise: e.g. even in the limit of vanishing inertia these studies can be useful for enhanced cooling in micro-devices. The main theme, however, was seeking optimal ways to build reduced order models in which to develop control strategies. Substantial progress has been driven by a large variety of newly available datasets, coupled with the emergence of new machine learning capabilities.

GAFD Quarter: Historically, collaborations between mathematicians, geoscientists and physicists began in the early 1920s when Lewis Fry Richardson of the UK Met Office pioneered the incorporation of numerical methods into weather forecasts.

Since that time, methods in applied mathematics have continued to play a critical role in understanding turbulence in astrophysical and geophysical fluid dynamics (GAFD), including recent advances in data assimilation techniques in forecasting models. The GAFD quarter convened a broad array of researchers from different backgrounds to address turbulence-related phenomena in planetary systems.



Figure 1: Local Rossby number in the Gulf Stream region, simulated with the ocean general circulation model ICON-O at a grid resolution of 600m. Local Rossby numbers near unity indicate that sub-mesoscale dynamics are resolved. (From the presentation by Peter Korn.)

Of particular interest was the workshop Advances in Geophysical and Astrophysical Turbulence (TURW05 May 16th-20th). Several presentations addressed pressing problems related to the Earth's weather and climate, such as dynamical processes that include phase changes of water and extreme heat events. The workshop also addressed turbulence in wider solar and planetary settings; for instance, in solar convection and geo-dynamo theory. The format included invited talks, as well as a poster session to provide visibility for early-career scientists.

Common themes emerged during the workshop which included the derivation of asymptotic equations for multi-scale phenomena, statistical theories for the interaction of turbulence with mean flows and the establishment and loss of balance. Connections to previous workshops were also present throughout, such as the issues of well-posedness and regularity of the fluid equations, the interplay of coherent structures and randomness and numerical approximation methods.

A final discussion identified promising directions for continued research such as smart numerical models which automatically adapt to asymptotic and multiscale regimes, the rigorous treatment of boundary conditions and the mathematical development of methods that extract and utilize information from sparse data. These are typically measured in geophysical and astrophysical settings.



Figure 2: Scattering of inertia-gravity waves by geostrophic turbulence. Top: Evolution of wave vertical velocity at fixed height in the simulation domain. Bottom: In wavenumber space, energy is scattered along the constant-frequency cone indicated by the stripes. (From the presentation by Jacques Vanneste.)

An example of continuing cross-cultural interaction among the "quarters" is the continuing collaboration between Rupert Klein (FUB), Leslie Smith and Sam Stechmann (climate scientists, both Wisconsin) with analysts Edriss Titi and Xin Liu (both Cambridge). Moreover, inspired by results from Baylor Fox-Kemper's global

ocean model, the three analysts Anna Mazzucato (PSU), Helena Nussenzveig-Lopes and Milton Lopes (both Rio de Janeiro) have begun a collaboration on the application of 2D Euler vortex methods to the ocean circulation.