Negative curvature and rigidity for von Neumann algebras

Approximation, deformation, quasification

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Rigidity theorems

What type of rigidity?

Theorems where

- weakly isomorphic structures are shown to be strongly isomorphic ;
- coarse structure is shown to determine much finer structure.

Examples:

- quasi-isometry of groups vs. virtual isomorphism of groups,
- homotopy equivalence vs. homeomorphism.

Our theme: von Neumann algebras.

- Discrete group Γ vs. group von Neumann algebra $L(\Gamma)$.
- Group action $\Gamma \curvearrowright (X, \mu)$ vs. crossed product $L^\infty(X) \rtimes \Gamma$.

Method: Popa's deformation/rigidity theory.

Focus today: usage of negative curvature phenomena.

Introduction to von Neumann algebras

Definition

A **von Neumann algebra** M is a weakly closed $*$-subalgebra of $B(\mathcal{H})$, the bounded operators on a Hilbert space.

- Weak topology: $T_i \to T$ iff $\langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$.

Example: group von Neumann algebra $L(\Gamma)$.

- Left regular representation: $\lambda : \Gamma \to \mathcal{U}(\ell^2(\Gamma)) : \lambda_g \delta_h = \delta_{gh}$.
- Then, $L(\Gamma)$ is the weak closure of $\mathbb{C}[\Gamma] = \text{span}\{\lambda_g \mid g \in \Gamma\}$.

Rigidity question: which information on Γ can be retrieved from $L(\Gamma)$?

Connes' theorem: all amenable icc groups Γ have isomorphic $L(\Gamma)$.

But: amenability, property (T), Haagerup property,... are invariants of $L(\Gamma)$.

Ozawa's solidity theorem

Some terminology first:

- a $\mathbf{II_1}$ **factor** is a von Neumann algebra M with trivial center and a tracial state $\tau : M \to \mathbb{C}$, i.e. $\tau(xy) = \tau(yx)$.
- $L(\Gamma)$ is a II_1 factor for any icc discrete group Γ.

Theorem (Ozawa, 2003)

For every icc **hyperbolic** group Γ, the II_1 factor $M = L(\Gamma)$ is **solid :**

whenever $A \subset M$ is not "locally trivial", we have that $A' \cap M$ is amenable.

- Notation: $A' \cap M = \{x \in M \mid ax = xa \text{ for all } a \in A\}$.
- Locally trivial: there is a projection $p \in A$ with $pAp = \mathbb{C}p$.
- Consequence: M is **prime**. We cannot write $M \cong M_1 \mathbin{\overline{\otimes}} M_2$.
  Note: $L(\Gamma_1 \times \Gamma_2) \cong L(\Gamma_1) \mathbin{\overline{\otimes}} L(\Gamma_2)$.

Why hyperbolic groups: Ozawa's class $\mathcal{S}$

Trivial observation: hyperbolicity leaves no room to commute.

Ozawa's class $\mathcal{S}$

We say that Γ belongs to class $\mathcal{S}$ if

- the left-right action $\Gamma \times \Gamma \curvearrowright \Gamma$ can be compactified to an action $\Gamma \times \Gamma \curvearrowright K$ by homeomorphisms of a compact space K (with $\Gamma \subset K$ densely),
- such that the right action $\{e\} \times \Gamma \curvearrowright K$ is trivial on the **boundary** $K \setminus \Gamma$,
- and such that the left action $\Gamma \times \{e\} \curvearrowright K$ is **amenable.**

If Γ is hyperbolic with Gromov boundary $\partial\Gamma$, take $K = \Gamma \cup \partial\Gamma$.

Negative curvature: the amenability of the boundary action $\Gamma \curvearrowright K$.

Class $\mathcal{S}$ and quasi-cocycles

Theorem (Ozawa, 2011)

A discrete group Γ belongs to class $\mathcal{S}$ if and only if

- Γ is **exact** (a technical condition that is "always" satisfied),
- there exists a proper $c : \Gamma \to \ell^2(\Gamma)$ such that for every $g \in \Gamma$, $\sup_{h\in\Gamma} \|c(gh) - \lambda_g c(h)\|_2 < \infty$.

Quasi-cocycle: $\sup_{g,h\in\Gamma} \|c(gh) - c(g) - \lambda_g c(h)\|_2 < \infty$.

So, class $\mathcal{S}$ is intimately related to

- Thom's class of groups with a nontrivial quasi-cocycle $c : \Gamma \to \ell^2(\Gamma)$,
- Monod-Shalom's class where $H_b^2(\Gamma, \ell^(\Gamma)) \neq \{0\}$,

with **non-elementary hyperbolic groups** being in the intersection of all.

Rigidity theorems

Recall: if Γ is nonamenable and in class $\mathcal{S}$, then $L(\Gamma)$ is prime.

This means: $L(\Gamma) \ncong M_1 \overline{\otimes} M_2$.

Theorem (Ozawa-Popa, 2003)

If Γ_i and Λ_j are nonamenable groups in class $\mathcal{S}$ and

$L(\Gamma_1 \times \cdots \times \Gamma_n) \cong L(\Lambda_1 \times \cdots \times \Lambda_m)$, then

$n = m$ and after reordering $L(\Gamma_i) \cong L(\Lambda_i)^{t_i}$.

Theorem (Chifan-de Santiago-Sinclair, 2015)

If Γ_i are nonelementary icc hyperbolic groups and $L(\Gamma_1 \times \cdots \times \Gamma_n) \cong L(\Lambda)$ for an **arbitrary** countable group Λ,

then $\Lambda \cong \Lambda_1 \times \cdots \times \Lambda_n$ and $L(\Gamma_i) \cong L(\Lambda_i)^{t_i}$.

$\leadsto$ For this last theorem, assuming Γ_i in class $\mathcal{S}$ is not sufficient. This is discussed later.

Group measure space construction (Murray - von Neumann)

Data : a countable group Γ acting on a probability space (X, μ), preserving μ.

Output : a tracial von Neumann algebra $M = L^\infty(X) \rtimes \Gamma$,

- generated by a copy of $A = L^\infty(X)$ and unitaries $(u_g)_{g \in \Gamma}$,
- dense $*$-algebra of finite sums $\sum_g a_g u_g$ with $a_g \in A$,
- product $au_g \, bu_h = a\alpha_g(b)\, u_{gh}$ where $\alpha_g(a) = a(g^{-1} \cdot \,)$ for $a, b \in A$,
- trace $\tau\left(\sum a_g u_g\right) = \int_X a_e \, d\mu$.

Silly remark : $L(\Gamma) = L^\infty(\{*\}) \rtimes \Gamma$.

Example : Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$.

Example : $\mathrm{SL}(n, \mathbb{Z}) \curvearrowright \mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$.

Freeness, ergodicity and Cartan subalgebras

Given $\Gamma \curvearrowright (X, \mu)$, write $A = L^\infty(X)$. Then $A \rtimes \Gamma$ is a II_1 factor if

- the action is **free :** almost every $x \in X$ has trivial stabilizer ;
- the action is **ergodic :** all Γ-invariant functions are a.e. constant.

In that case, $A \subset A \rtimes \Gamma$ is a **Cartan subalgebra.**

$\rightsquigarrow$ A Cartan subalgebra $A \subset M$ is a maximal abelian subalgebra such that $\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$ generates M.

Singer's theorem : the inclusion $A \subset A \rtimes \Gamma$ contains the same data as the **orbit equivalence relation** $\mathcal{R}(\Gamma \curvearrowright X) = \{(x, y) \mid x \in \Gamma \cdot y\}$.

$\rightsquigarrow$ If one can prove **uniqueness of the Cartan subalgebra,** then this equivalence relation can be recovered from the II_1 factor.

Absence / uniqueness of Cartan subalgebras

- (Voiculescu, 1995) The free group factors $L(\mathbb{F}_n)$ have no Cartan subalgebra. Method: free probability theory.
- (Ozawa, 2008) The group $\Gamma = \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ belongs to class $\mathcal{S}$.

  Then, $L(\mathbb{Z}^2) \subset L(\Gamma)$ is a Cartan subalgebra.

  Conclusion: class $\mathcal{S}$ is not enough to prove absence of Cartan for $L(\Gamma)$ or uniqueness of Cartan for $L^\infty(X) \rtimes \Gamma$.

Theorem (Ozawa-Popa, 2007-2008)

Assume that Γ is nonamenable and that

- Γ admits a proper cocycle $c : \Gamma \to \ell^2(\Gamma)$, (strengthens class $\mathcal{S}$)
- Γ has the complete metric approximation property. (extra condition)

Then, $L(\Gamma)$ has no Cartan subalgebra.

If $\Gamma \curvearrowright (X,\mu)$ is free ergodic and **profinite,** then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy.

About the Ozawa-Popa theorem, part 1

Unique Cartan in $L^\infty(X) \rtimes \Gamma$ for **profinite** actions of nonamenable Γ

- that admit a proper cocycle $c : \Gamma \to \ell^2(\Gamma)$, (strengthens class $\mathcal{S}$)
- that have the complete metric approximation property. (extra condition)

Definition

A **Fourier multiplier** on Γ is a map $\varphi : \Gamma \to \mathbb{C}$ such that the multiplier $m_\varphi : L(\Gamma) \to L(\Gamma) : m_\varphi(u_g) = \varphi(g)u_g$ is **completely bounded.**

- Positive-definite functions are Fourier multipliers with $\|m_\varphi\|_{\mathrm{cb}} = \varphi(e)$.
- **Amenability:** there exist finitely supported positive-definite $\varphi_n : \Gamma \to \mathbb{C}$ such that $\varphi_n \to 1$ pointwise.
- **CMAP:** there exist finitely supported Fourier multipliers $\varphi_n : \Gamma \to \mathbb{C}$ such that $\varphi_n \to 1$ pointwise and $\limsup_n \|\varphi_n\|_{\mathrm{cb}} = 1$.
- **Weak amenability:** there exist finitely supported Fourier multipliers $\varphi_n : \Gamma \to \mathbb{C}$ such that $\varphi_n \to 1$ pointwise and $\limsup_n \|\varphi_n\|_{\mathrm{cb}} < \infty$.

About the Ozawa-Popa theorem, part 2

Definition

A probability measure preserving (pmp) action $\Gamma \curvearrowright (X,\mu)$ is **profinite** if it is the inverse limit of a system of pmp actions $\Gamma \curvearrowright (X_n,\mu_n)$ where each X_n is a finite set.

- If Γ has CMAP and $\Gamma \curvearrowright (X,\mu)$ is profinite, then the von Neumann algebra $M = L^\infty(X) \rtimes \Gamma$ has CMAP.
- Ozawa-Popa, 2007: given a II_1 factor M with CMAP and given an amenable $A \subset M$, something happens.
- Ozawa, 2010: if G is weakly amenable and $N \lhd G$ is an amenable normal subgroup, then N admits a mean that is invariant under translation by N and under conjugation by G.
- Something similar in the context of $A \subset M$, yielding a $\mathcal{N}_M(A)$-invariant mean "somewhere".

CMAP and weak amenability, Cowling's conjecture

- Groups with **CMAP:** free groups $\mathbb{F}_n$, lattices in $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$.

  Stable under direct products, free products and measure equivalence.

- **Weakly amenable** groups: lattices in $\mathrm{Sp}(n,1)$, hyperbolic groups.

  Stable under direct products and measure equivalence.

  Open problem: stability under free products.

Haagerup property: existence of a proper isometric action of Γ on a Hilbert space.

Cowling's conjecture: Haagerup property iff CMAP.

- Direction $\Rightarrow$ was disproved by CoStVa-OzPo using e.g. $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{F}_2$.
- Direction $\Leftarrow$ remains open.

Uniqueness of Cartan subalgebras

Theorem (Chifan-Sinclair, 2011)

Assume that

- Γ belongs to class $\mathcal{S}$,
- Γ is weakly amenable, but nonamenable.

Then, $L(\Gamma)$ has no Cartan subalgebra.

If $\Gamma \curvearrowright (X, \mu)$ is free ergodic and **profinite,** then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \Gamma$, up to unitary conjugacy.

$\rightsquigarrow$ Again, profiniteness ensures $L^\infty(X) \rtimes \Gamma$ weakly amenable.

Theorem (Popa-V, 2011-2012)

Exactly the same result, but for **arbitrary** free ergodic probability measure preserving actions $\Gamma \curvearrowright (X, \mu)$.

$\rightsquigarrow$ Weak amenability has to be exploited **“relative to $L^\infty(X)$”.**

Crossed products with free groups

Corollary (Popa-V, 2011)

If $\mathbb{F}_n \curvearrowright X$ and $\mathbb{F}_m \curvearrowright Y$ are free ergodic pmp actions and if $L^\infty(X) \rtimes \mathbb{F}_n \cong L^\infty(Y) \rtimes \mathbb{F}_m$, then $n = m$.

- By uniqueness of Cartan, an isomorphism maps $L^\infty(X)$ onto $L^\infty(Y)$.
- It thus induces an **orbit equivalence** of the actions : isomorphism $\Delta : X \to Y$ with $\Delta(\mathbb{F}_n \cdot x) = \mathbb{F}_m \cdot \Delta(x)$ for a.e. $x \in X$.
- This implies that $n = m$ by one of Gaboriau's invariants for countable equivalence relations : cost or the first L^2-Betti number.

Note: it is wide open whether $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ when $n \neq m$.

Cartan rigidity

Definition

A discrete group Γ is called **Cartan-rigid** if for all free ergodic pmp $\Gamma \curvearrowright (X, \mu)$, the crossed product $L^\infty(X) \rtimes \Gamma$ has a unique Cartan subalgebra up to unitary conjugacy (namely, $L^\infty(X)$).

The following groups are Cartan-rigid:

- (Popa-V, 2012) nonamenable, weakly amenable groups in class $\mathcal{S}$, in particular nonelementary hyperbolic groups,
- (Ioana, 2012) all free products $\Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$.
- a "natural strengthening" of Cartan-rigidity is stable under extensions.

In principle: any property that "strongly excludes" normalish abelienish subgroups might imply Cartan-rigidity.

Open problem: is $\mathrm{SL}(n, \mathbb{Z})$ Cartan-rigid for $n \geq 3$?

Crossed products with locally compact groups

- Weak amenability and Ozawa's class $\mathcal{S}$ are well defined for **locally compact** groups.
- All rank one connected simple Lie groups with finite center have these properties.
- Also: all groups that act properly on a tree, or that act properly on a hyperbolic graph.

Theorem (Brothier-Deprez-V, 2017)

Let G be a weakly amenable **locally compact** group in Ozawa's class $\mathcal{S}$.
Let $G \curvearrowright (X, \mu)$ be any free, ergodic, nonsingular action.

- Either the action is amenable in Zimmer's sense, equivalently $L^\infty(X) \rtimes G$ is amenable.
- Or $L^\infty(X) \rtimes G$ has a unique Cartan subalgebra.

Cross section equivalence relations

Attention: $L^\infty(X)$ is not a Cartan subalgebra in $L^\infty(X) \rtimes G$.

- A Borel set $X_1 \subset X$ is called a **cross section**

  if there exists a neighborhood $\mathcal{U} \subset G$ of e such that the map $\mathcal{U} \times X_1 \to X : (g, x) \mapsto g \cdot x$ is injective with non-null range.

- Then, $\mathcal{R} = \{(x, x') \in X_1 \times X_1 \mid x \in G \cdot x'\}$ is a **countable** nonsingular equivalence relation.

- Up to amplification, this produces the canonical Cartan subalgebra of $L^\infty(X) \rtimes G$.

W* strong rigidity theorem

Theorem (Brothier-Deprez-V, 2017)

Let G_1, G_2, H_1, H_2 all be weakly amenable, nonamenable and in class $\mathcal{S}$. Assume unimodularity.

Let $G_1 \times G_2 \curvearrowright (X, \mu)$ and $H_1 \times H_2 \curvearrowright (Y, \eta)$ be free ergodic pmp actions.

If $L^\infty(X) \rtimes (G_1 \times G_2) \cong L^\infty(Y) \rtimes H_1 \times H_2$, then

the actions are **conjugate:** $\Delta : X \to Y$ and $\delta : G_1 \times G_2 \to H_1 \times H_2$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$.

Method:

- Uniqueness of the Cartan subalgebra produces an orbit equivalence.
- Orbit equivalence rigidity à la Monod-Shalom and Sako.