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I'LL GIVE YOU Big Proof
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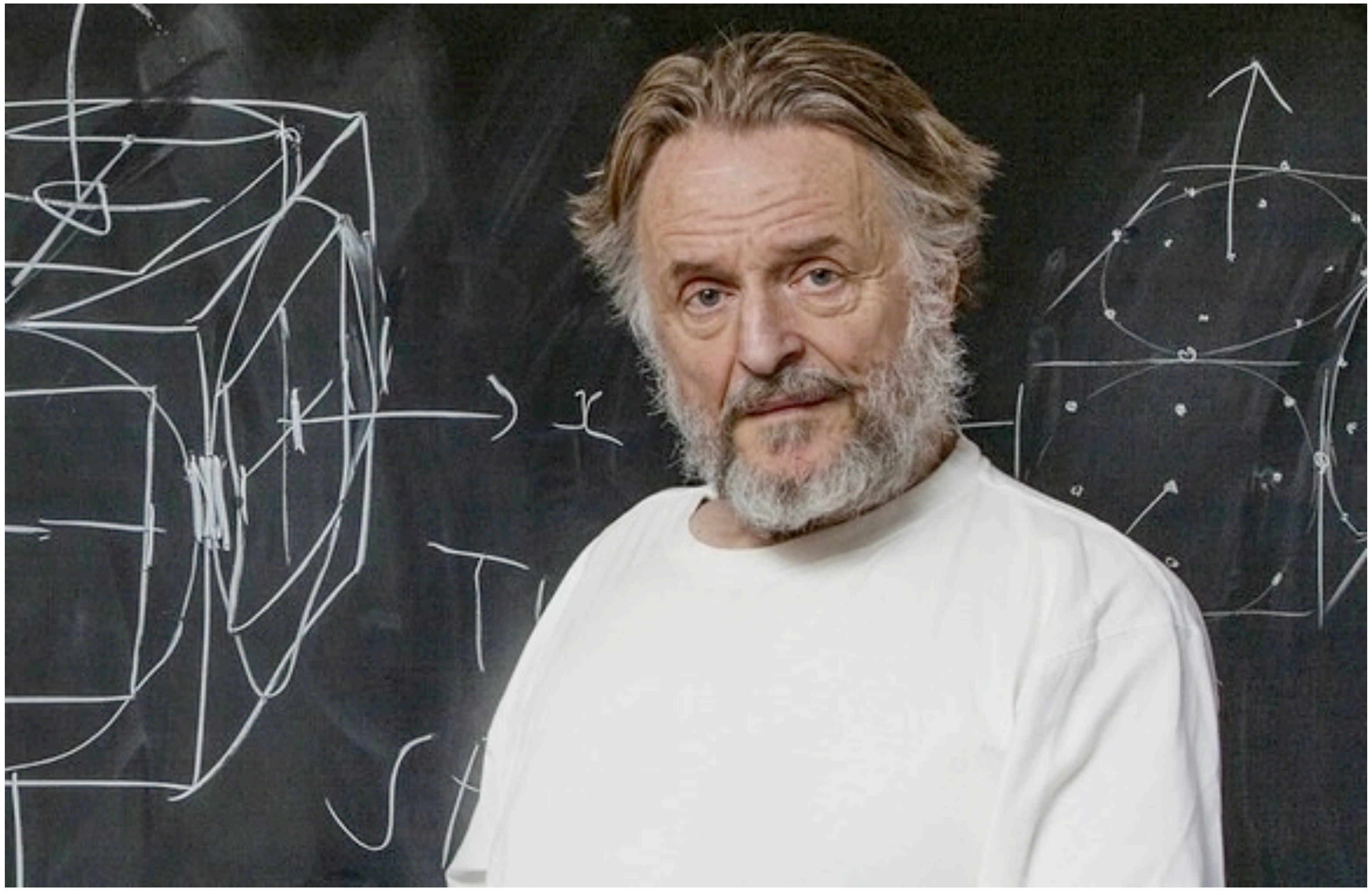
Big Conjecture

Thomas Hales

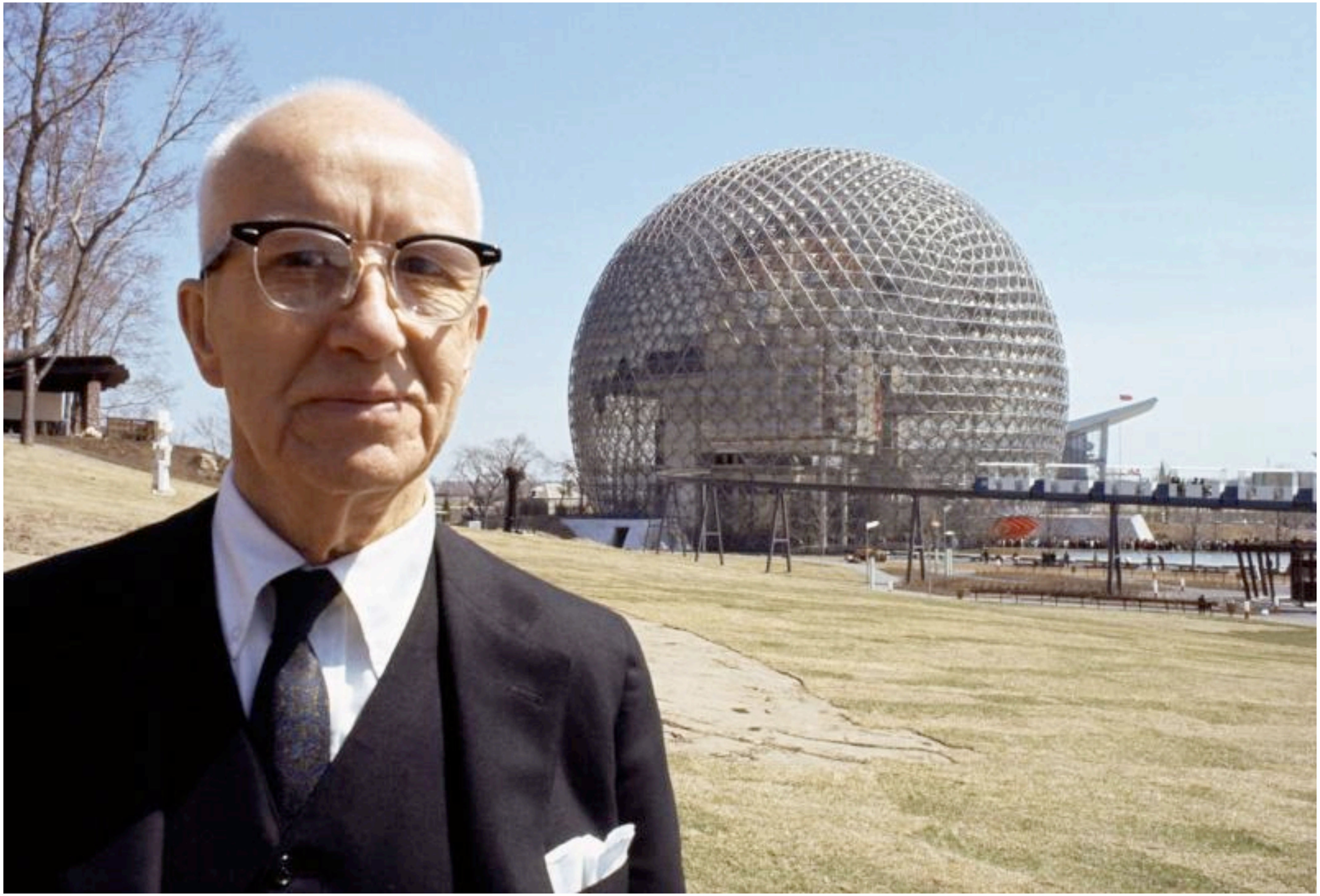
July 10, 2017

Everything is vague to a degree you do not realize till you have tried to make it precise. – Bertrand Russell

Keep your conjectures bold and your refutations brutal. – Nick Horton



Sphere Packings



The Mathematical Intelligencer

We're Not Afraid Of Controversy...

We Welcome It!

The Mathematical Intelligencer has long been the main forum for debate between some of the world's most renowned and respected mathematicians. **The Mathematical Intelligencer** has always provided a place for the debate of all mathematical issues. Inside you'll find just a few of the most notable controversies that **The Mathematical Intelligencer** has proudly published in the past, and some of the controversies you can look forward to in the future.

THE KEPLER CONJECTURE CONTROVERSY

Perhaps the most controversial topic to be covered in **The Mathematical Intelligencer** is the Kepler Conjecture. In **The Mathematical Intelligencer** (16:3), Thomas C. Hales takes on Wu-Yi Hsiang's 1990 announcement that he had proved the Kepler Conjecture, the conjecture that no arrangement of spheres of equal radius in 3-space has density greater than that of the face-centered cubic packing.

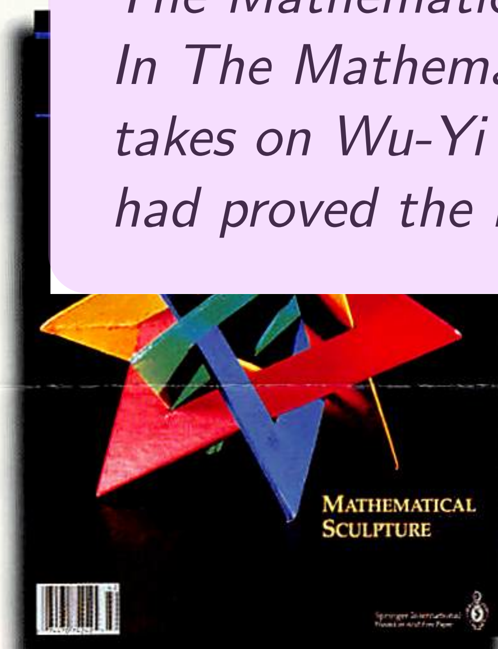
Following are excerpts from the article
"The Status of the Kepler Conje

Hsiang was honored for his work in January meetings of the AMS-MAA, by being inv plenary address entitled "The proof of Kepler's conjecture on the sphere-packing problem."

As a result of such announcements, many are prone to accept Hsiang's solution to the sphere-packing problem. Even if Hsiang withdraws his claims, some might continue to believe, for years to come, that the problem has been successfully solved. It has become necessary, therefore, to write this article on the status of the Kepler conjecture, to correct the public record.

What is the significance of this negative result? Hsiang's early preprint omitted the argument for seven-faced polyhedra; it merely remarked that "it is easy to see that no vertices...have more than six forks." (The number of forks is the number of edges or faces surrounding the vertex.) The fact that this much analysis was required to study a single arrangement shows that those who challenged his "easy to see" claim had more than ample justification for doing so. He claims to use deformation arguments, and deformation arguments (properly developed), even if linearized, require the solution to large systems of equations.

His packing bounds are dependent on this result. In later arguments he uses case-by-case arguments that list all relevant polyhedra with only four, five, or six faces around a given edge. Hence, we must put all his later conclusions on indefinite hold. One is left to conclude that his hasty reduction has no real substance to it and that his critical case remains an isolated test case.



In the end, I feel that Hsiang has missed the point of the subject of sphere packings. Many packing problems have geo-

implausible configurations could be dismissed without proof. But rigor requires that proofs be given.

One of the most unsettling aspects of his article is his deliberate and persistent use of methods that are known to be defective. The errors in his hole-fitting principle and his size-decreasing deformation were pointed out to him some time ago. His claims over the last 3 years that the next revision will answer all objections have grown tiresome.

In conclusion, I offer a suggestion. First, Hsiang should withdraw his claim to have resolved the Kepler conjecture. Mathematicians can easily spot the difference between handwaving and proof. Then, Hsiang should isolate the statements in his article that he was unable to prove rigorously. He should show carefully how the Kepler conjecture would follow from these statements. In this way, his work would make an important contribution to the field. It would provide a concrete program that could eventually lead to a solution to the problem. Instead, by presenting experimental hypothesis as fact, he destroys the credibility of his own work.

"Perhaps the most controversial topic to be covered in The Mathematical Intelligencer is the Kepler Conjecture. In The Mathematical Intelligencer, Thomas C. Hales takes on Wu-Yi Hsiang's 1990 announcement that he had proved the Kepler Conjecture,..."

HSIANG RESPONDS

Wu-Yi Hsiang has agreed to publish his rejoinder to Thomas Hales, and the

From hales@math.lsa.umich.edu Wed Aug 19 02:43:02 1998
Date: Sun, 9 Aug 1998 09:54:56 -0400 (EDT)
From: Tom Hales <hales@math.lsa.umich.edu>
To:

Subject: Kepler conjecture

Dear colleagues,

I have started to distribute copies of a series of papers giving a solution to the Kepler conjecture, the oldest problem in discrete geometry. These results are still preliminary in the sense that they have not been refereed and have not even been submitted for publication, but the proofs are to the best of my knowledge correct and complete.

Nearly four hundred years ago, Kepler asserted that no packing of congruent spheres can have a density greater than the density of the face-centered cubic packing. This assertion has come to be known as the Kepler conjecture. In 1900, Hilbert included the Kepler conjecture in his famous list of mathematical problems.

In a paper published last year in the journal "Discrete and Computational Geometry," (DCG), I published a detailed plan describing how the Kepler conjecture might be proved. This approach differs significantly from earlier approaches to this

June 2002 referee report (4 years in)

iii. Checking (and re-running) the program, which is working in Phase 3, might detect a "case" in which the mentioned function is negative. Then the theory would collapse (in its present form), and would require amendment, since the suggested decomposition of the space would not have the claimed property.

With all this in mind one would prefer to have Phase 2 and Phase 3 checked *prior to start working on Phase* 1 (and minimize the chance that the essential work of careful reading of the manuscript might prove useless). Since I am not planning to read any part of Phase 2 and/or 3, — and some other referees might share my views — I would like to ask you to inform me whether the Editorial Board has organized any separate proceedings regarding the checking of Phase 2 and 3 or no support of this kind can be expected.

Dear Thomas Hales,

Steven Obua has reported to you that I am working on the tame graphs generation part of the Kepler proof.

I have completed a translation of your Java programm to an executable Isabelle specification, and tested it agains the graphs generated with your programm. A proof of correctness of my Isabelle specification would also prove the correctness of the Java program. Furthermore the generated graphs can be used for the other parts of the proof.

What I have left out for the moment, is the calculation, if a graph is isomorphic to one of a stored set of graphs, so the output of my ML program is about a million of graphs.

Now I started with the verification. Could you please help me with some questions about the algorithm ? For the moment I have only two questions, but may I come with more questions later?

Thank you very much,

best regards,
Gertrud Bauer

1. what is the intuition for calling handleQuad only for "quadfriendly" graphs?

   whenever you want to refine a quad, it should be sufficient to generate all 1 + 1 + 2 + 4 quad combinations obtained from property tame4 ?

2. I am trying to verify the correctness of disregarding "neglectable graphs". Therefore I need to show that for all tame, final graphs,

   $faceQuanderLowerBound + ExcessNotAt <= target.$

   this is true for an exceptional parameter, since

Lessons from the early formalization of the Kepler conjecture:

- Do not be afraid of controversy.
- Formalization of partial results is useful (Bauer-Nipkow).
- Formalizers should be part of the *tactical response teams* that routinely intervene in prominent, difficult, unpublished proofs.

Examples where *tactical response teams* might have helped.

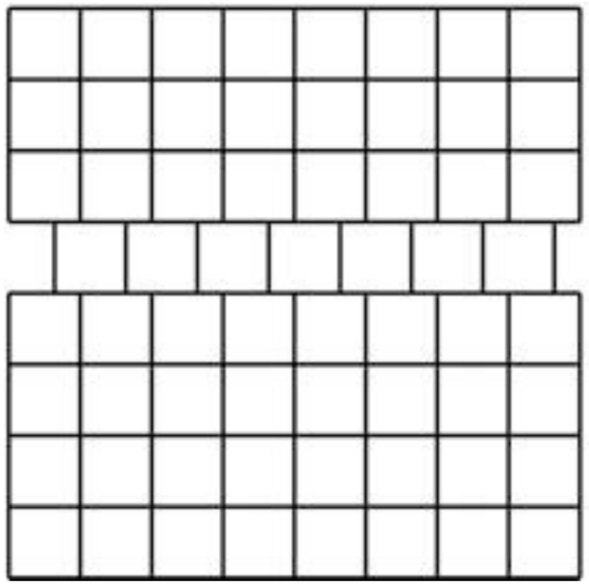
- ABC conjecture (5 years of review so far!)
- Poincaré conjecture (3 years of review)
- Hironaka's announcement.
- HMAC brawl in cryptography (a controversy of particular interest for the formal proof community, because it involves us). Should security proofs be allowed to make the assumption that cryptographic protocols withstand non-uniform adversaries?

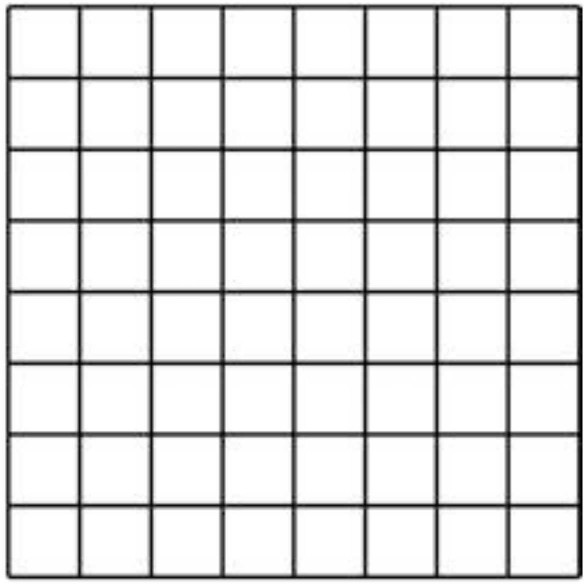
Maryna Viazovska has proved the Cohn-Elkies conjecture (2003). This solves the sphere packing problem in 8 and (in collaboration with others) in 24 dimensions.

The formalization of this result would be an great project for a graduate student. (It is a feasible project.) It would involve building formal libraries for Fourier analysis, Schwartz functions, the Laplace transform, modular forms, and a bit of interval arithmetic.

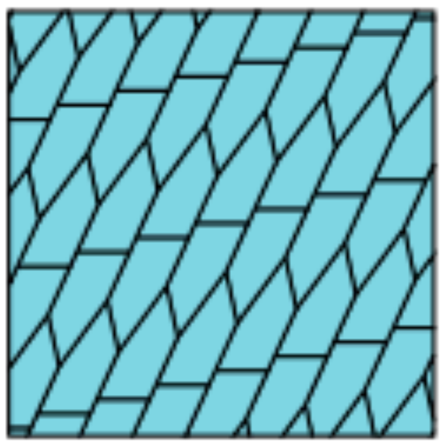
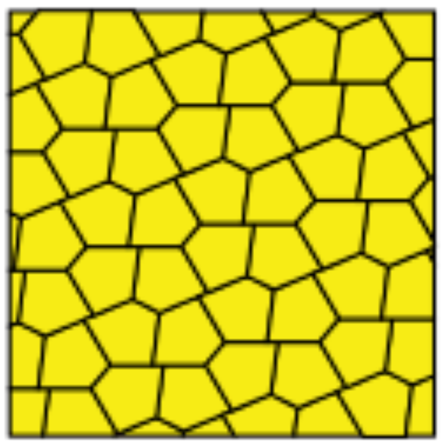
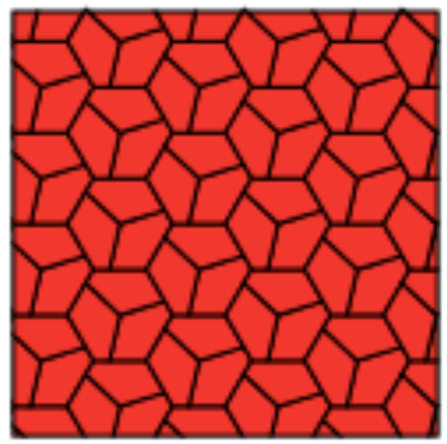
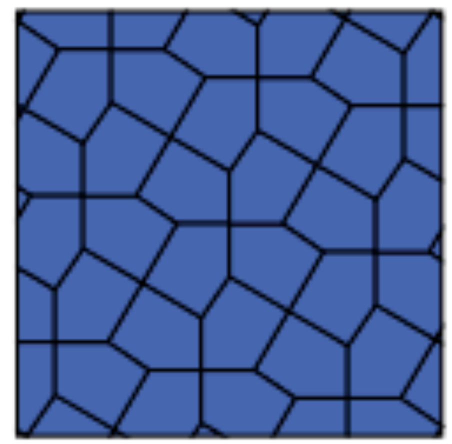
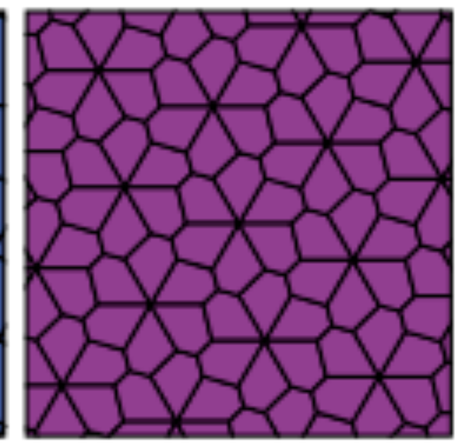
Convex Tiles

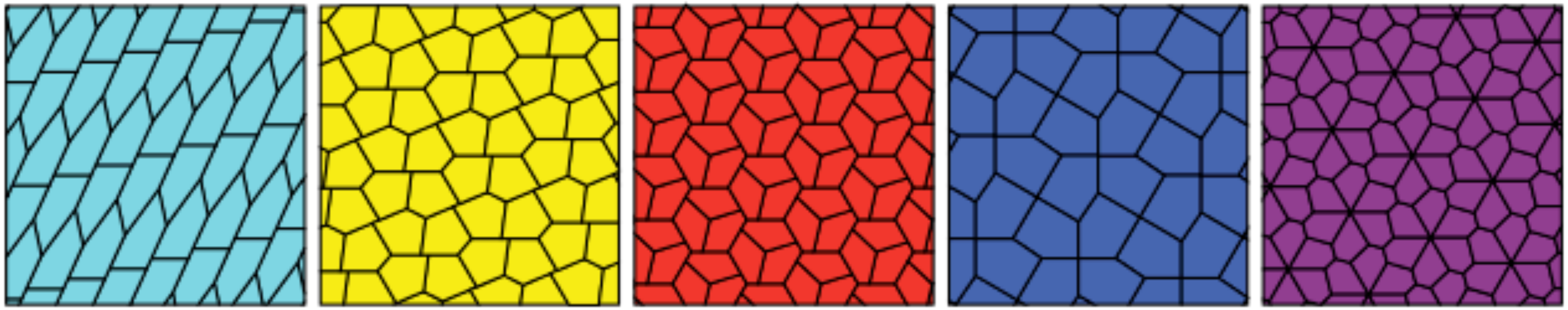
The third part of Hilbert's 18th problem asks whether anisohedral tiles exist.
The first anisohedral tile was constructed (in 3D) by Reinhardt in 1928.

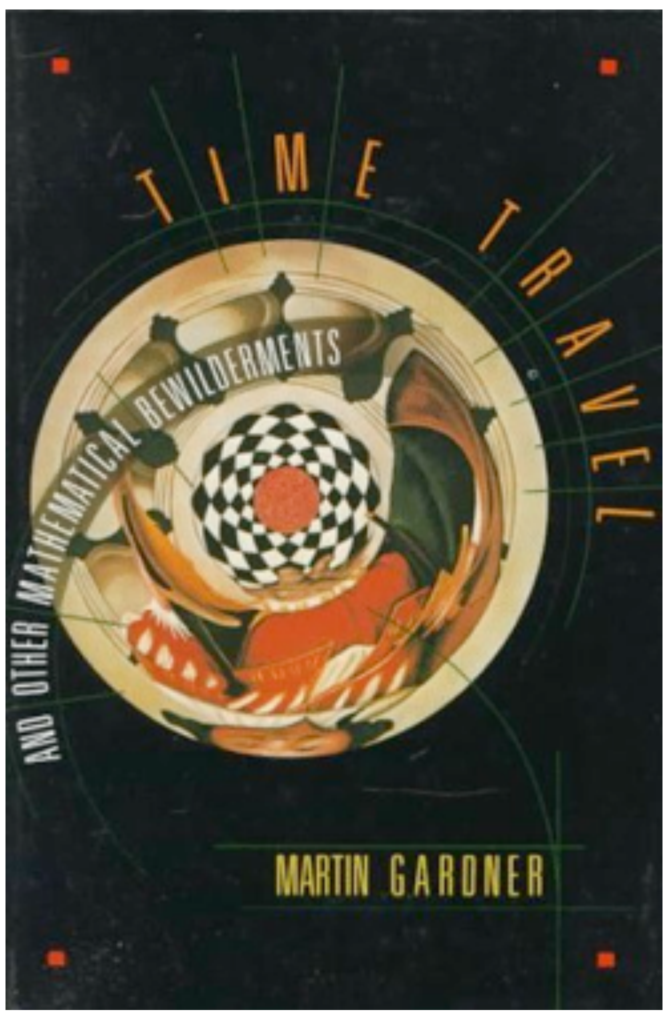




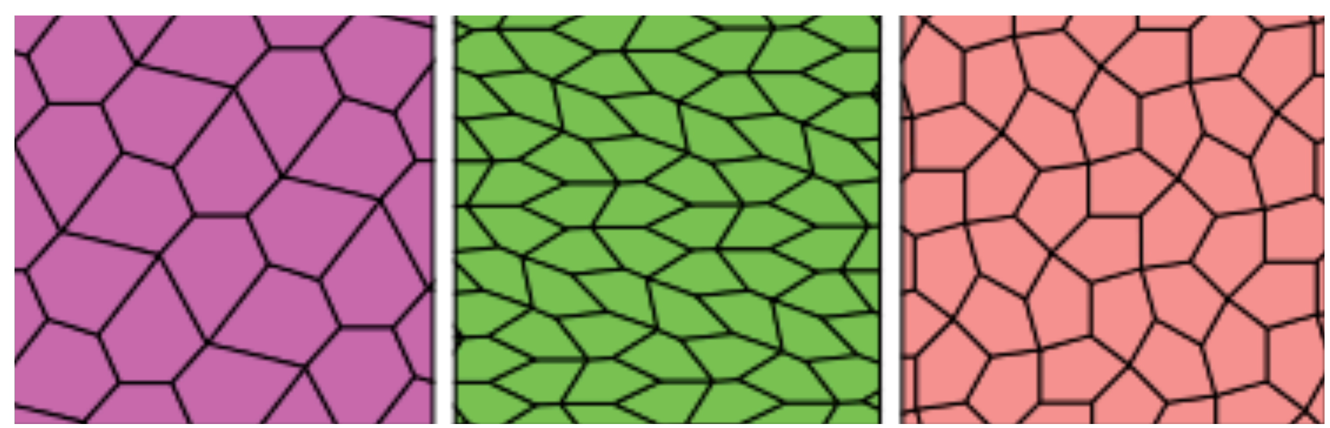
As a graduate student, Reinhardt attempted to classify all convex polyhedral tiles, as a test of Hilbert's problem. He classified them all except pentagons. This classification problem remained open for 99 years, until solved earlier this year by Michael Rao.

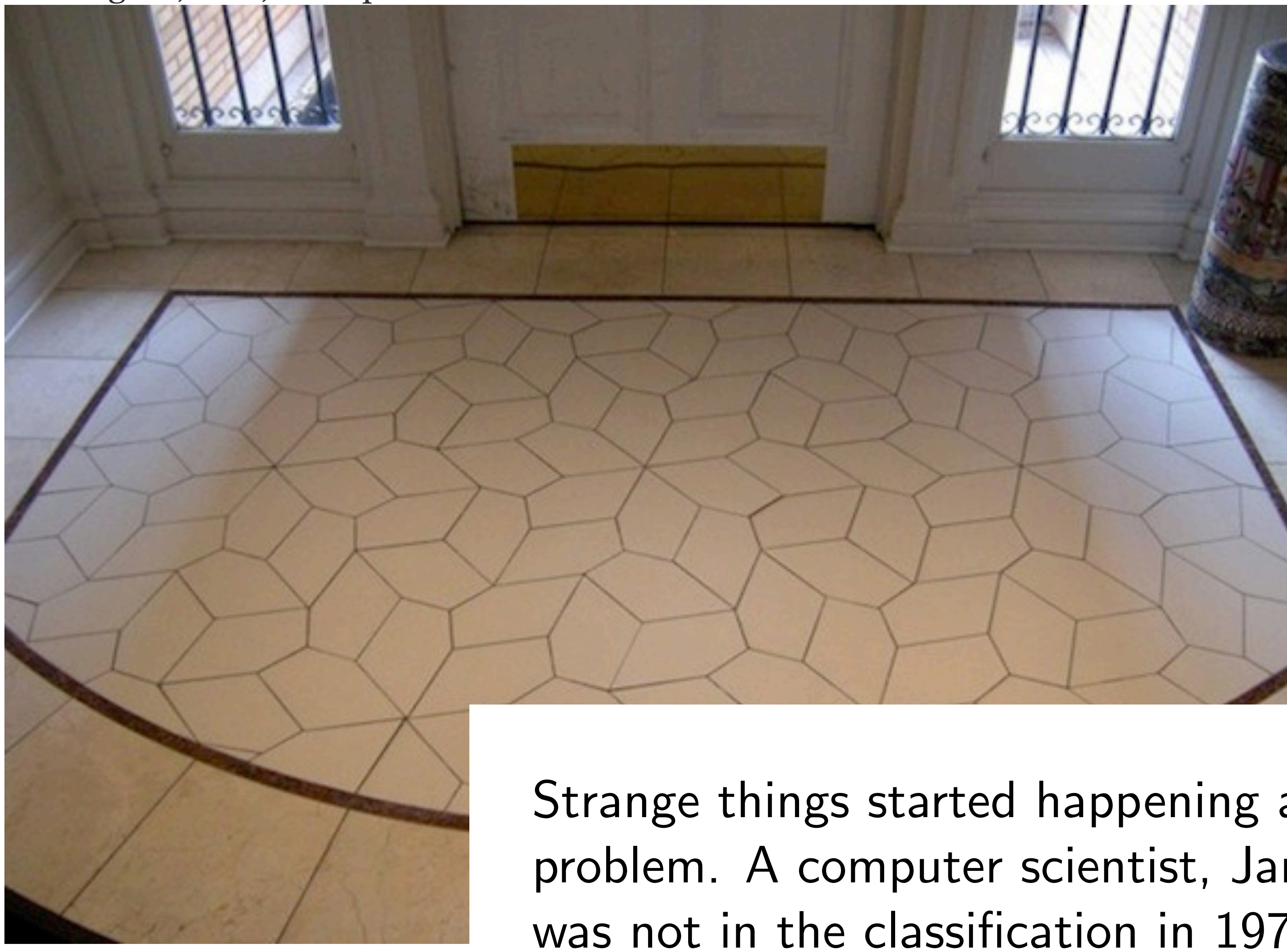




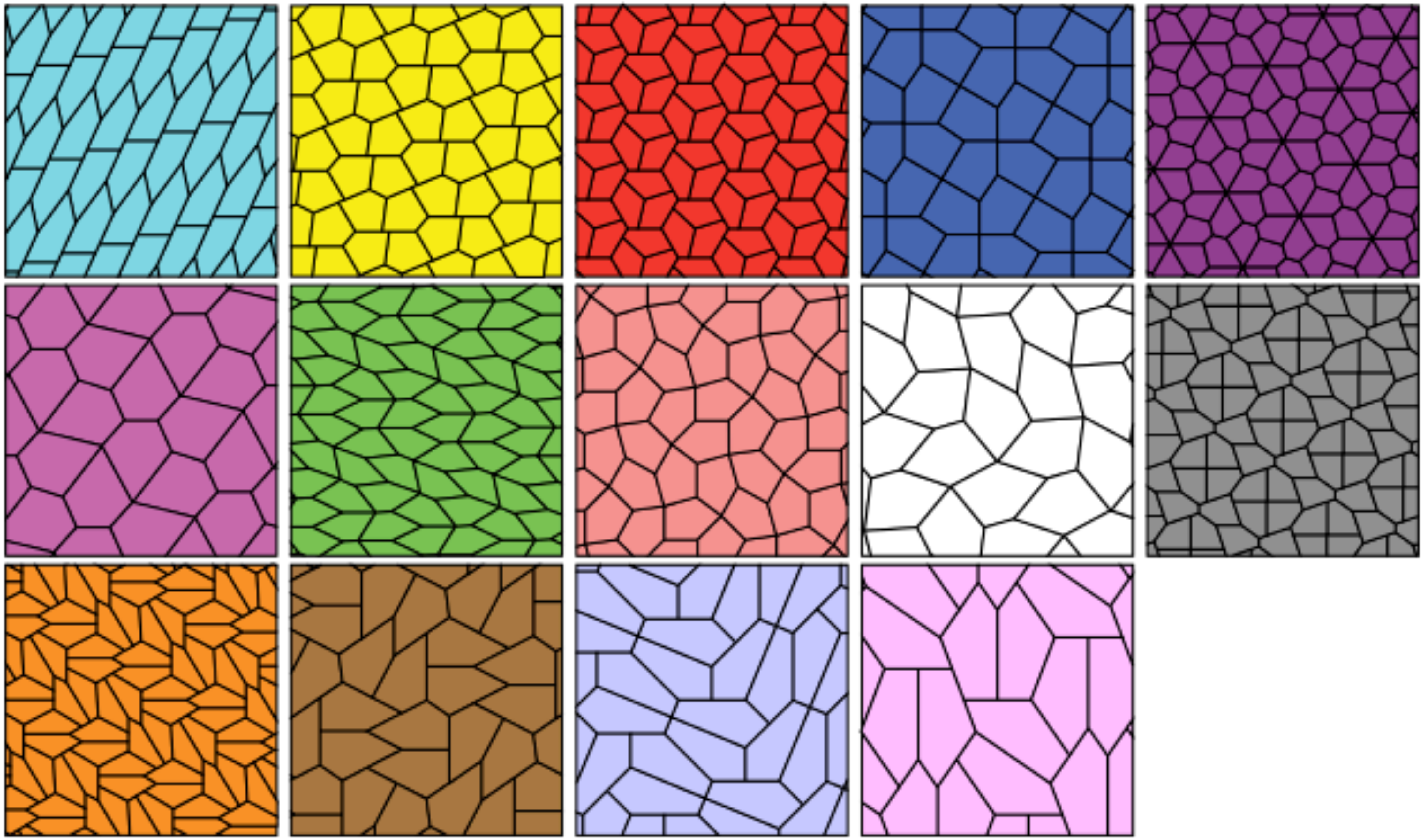
Kershner wrote, “the proof that the list. . . is complete is extremely laborious and will be given elsewhere.” That proof was never published. However, Kershner’s classification became widely known, when Martin Gardner popularized it in a column in *Scientific American*.

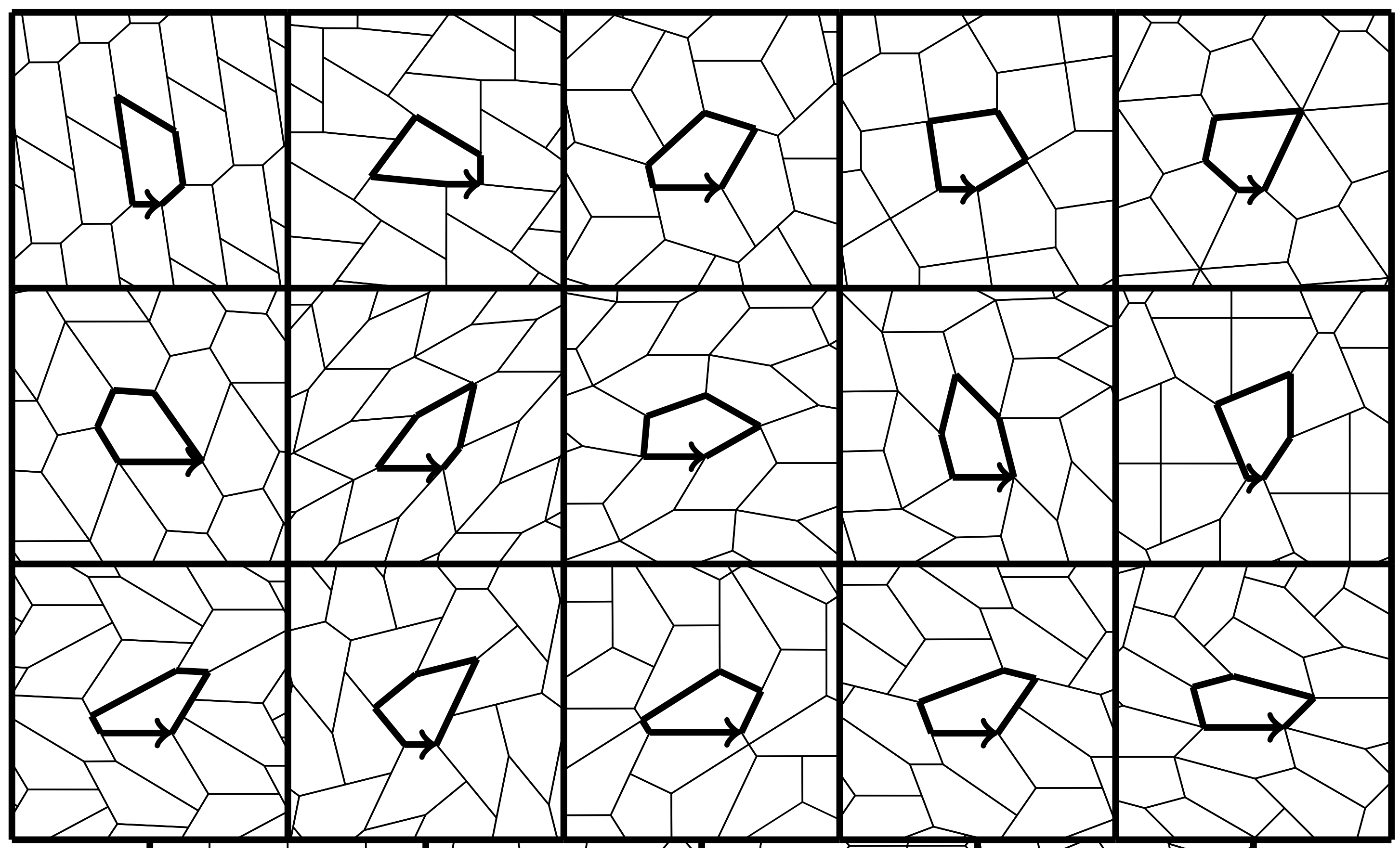


Washington, D.C., headquarters of the Mathematical Association of America



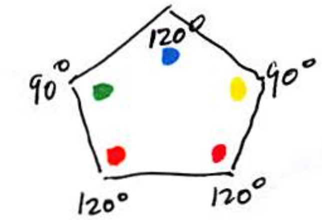
Strange things started happening after Gardner popularized the problem. A computer scientist, James, found a pentagon tile that was not in the classification in 1975. Four more tiles were soon found by an amateur mathematician, Rice. By 2015, a total of 15 families of convex pentagon tiles had been found. (The discovery of the 15th received widespread attention in the press, including National Public Radio.) In summary, not only was Kershner's proof incorrect, but the classification that he was trying to prove was entirely false.





Example

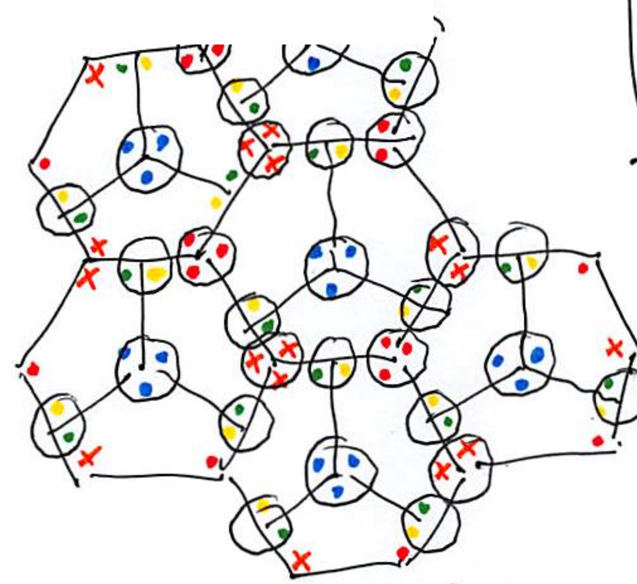
pentagon =



Rao's classification proof is computer assisted (about 5000 lines of C++).

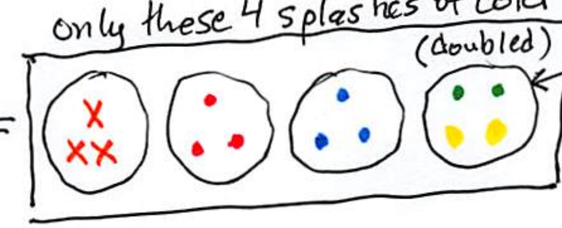
Rao has made the suggestion of making a formal proof in Coq or similar software.

Rao's classification gives 371 color palettes.



only these 4 splashes of color occur in this tiling.

Color palette of this tiling =



Note that we double the colors here because they form a ½-circle in the tiling

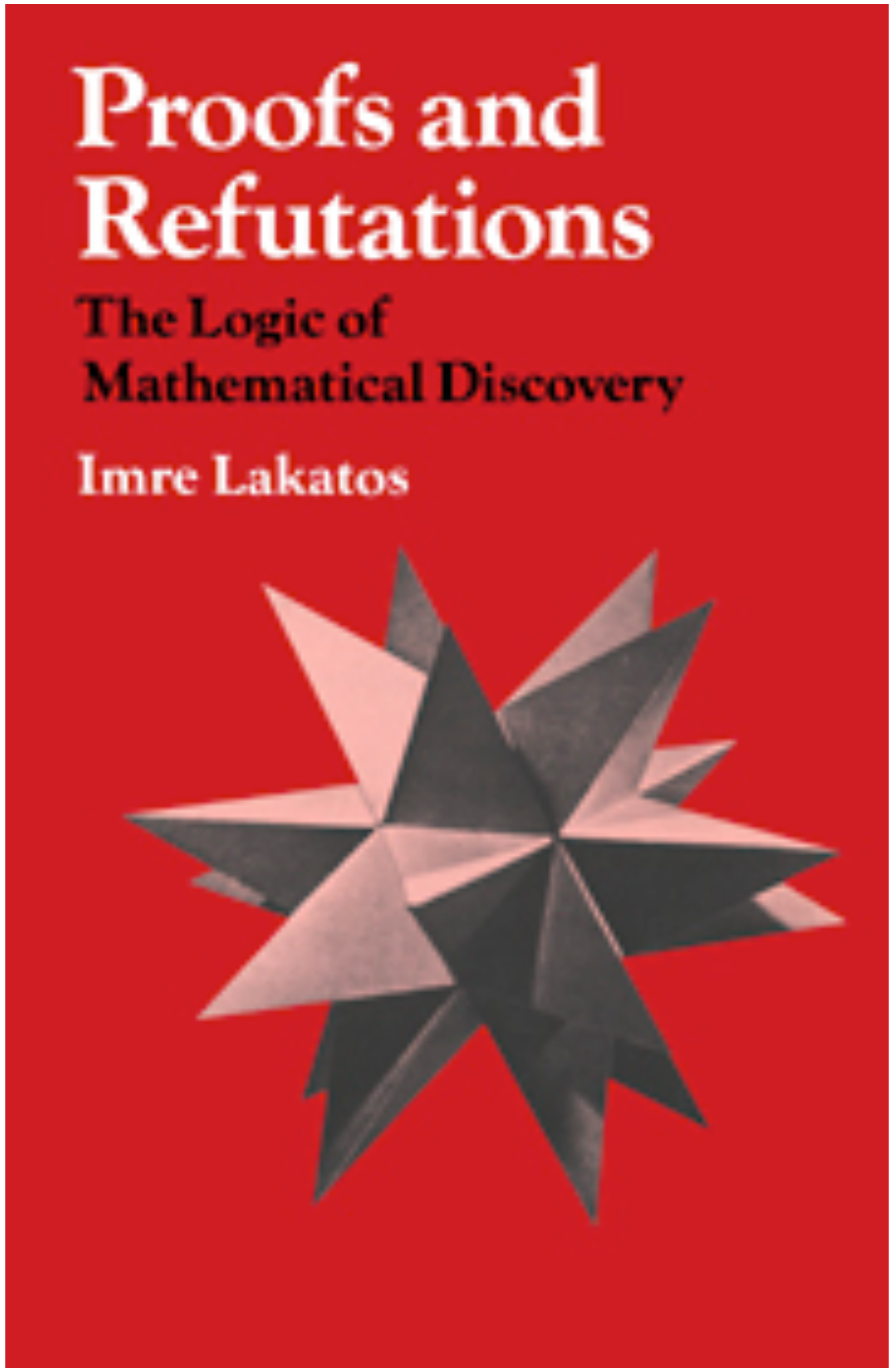
The palette determines the angles in this case as follows:

$$\textcolor{red}{\times} + \textcolor{red}{\times} + \textcolor{red}{\times} = 360 \Rightarrow \textcolor{red}{\times} = 120^\circ$$

$$\textcolor{blue}{\bullet} + \textcolor{blue}{\bullet} + \textcolor{blue}{\bullet} = 360 \Rightarrow \textcolor{blue}{\bullet} = 120^\circ$$

$$\textcolor{red}{\bullet} + \textcolor{red}{\bullet} + \textcolor{red}{\bullet} = 360 \Rightarrow \textcolor{red}{\bullet} = 120^\circ$$

$$\left.\begin{array}{l} \textcolor{green}{\bullet} + \textcolor{green}{\bullet} + \textcolor{yellow}{\bullet} + \textcolor{yellow}{\bullet} = 360 \\ \textcolor{red}{\times} + \textcolor{blue}{\bullet} + \textcolor{red}{\bullet} + \textcolor{green}{\bullet} + \textcolor{yellow}{\bullet} = 3(180) \end{array}\right\} \Rightarrow \cancel{\begin{array}{l} \textcolor{yellow}{\bullet} = 90^\circ \\ \textcolor{green}{\bullet} = 90^\circ \end{array}} \Rightarrow \boxed{\textcolor{yellow}{\bullet} + \textcolor{green}{\bullet} = 180^\circ}$$

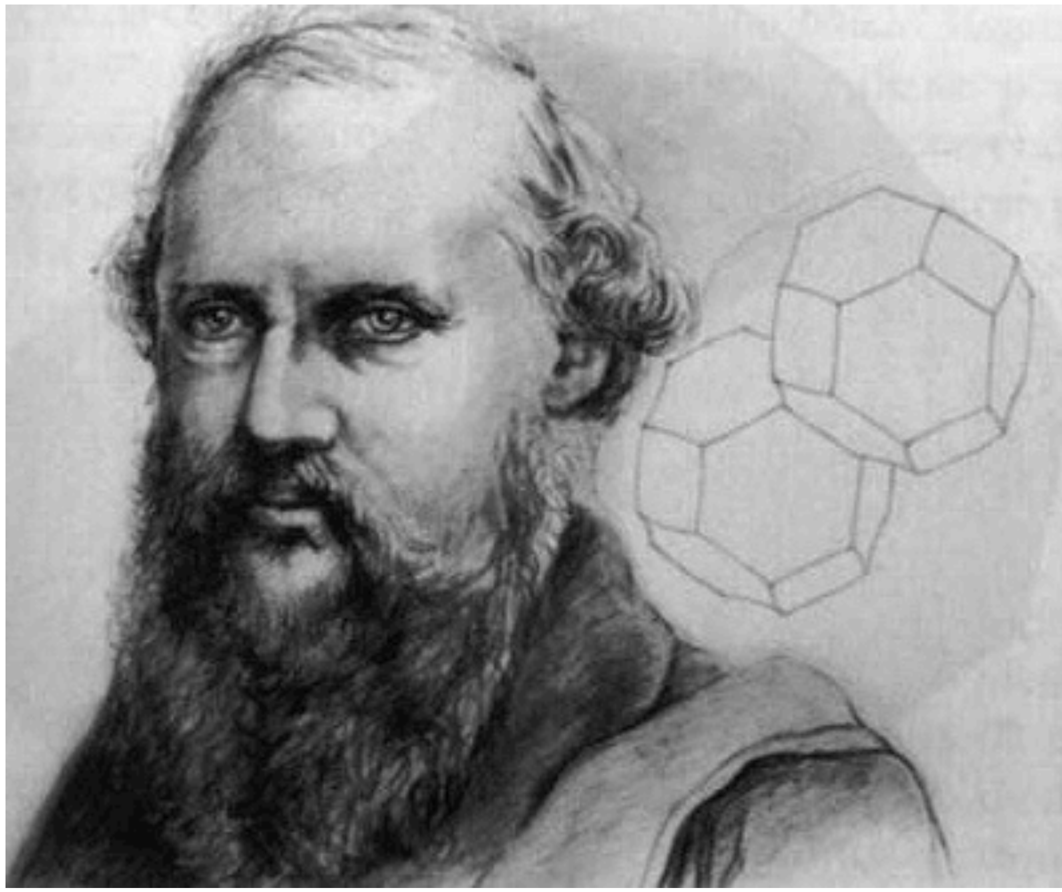
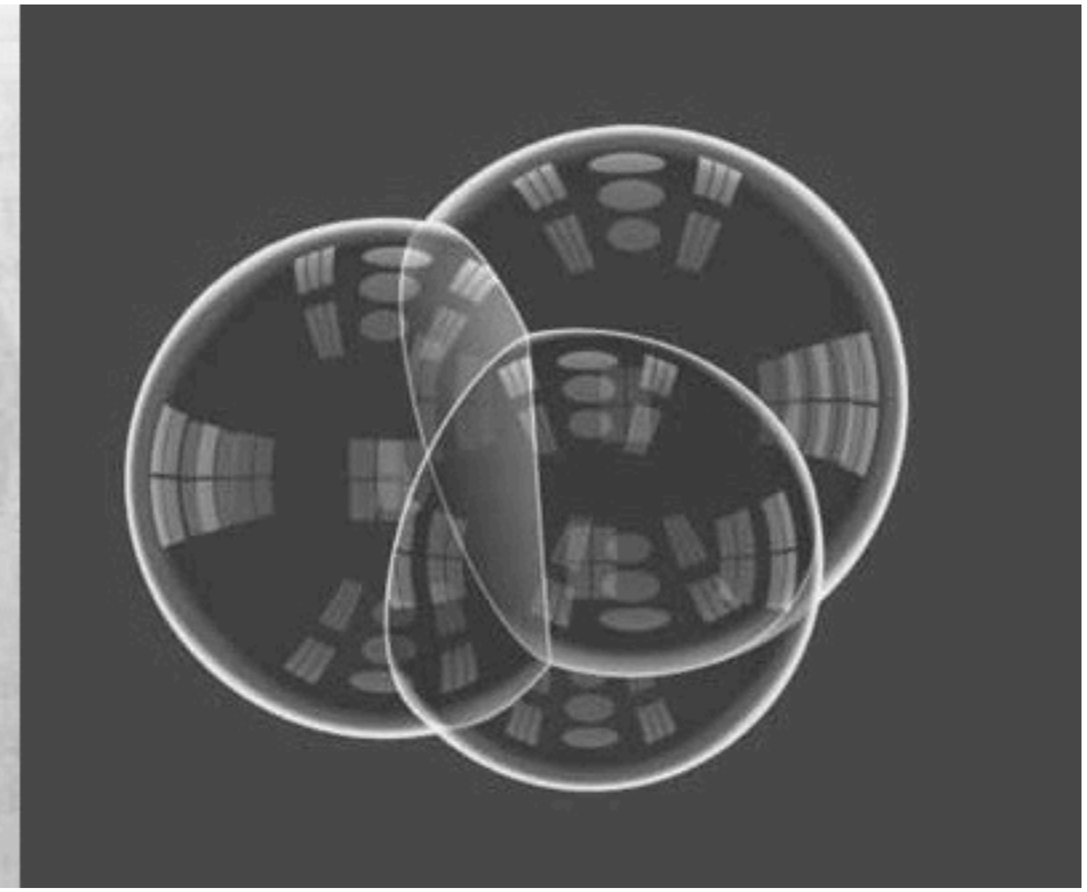

Proofs and Refutations
The Logic of Mathematical Discovery
Imre Lakatos

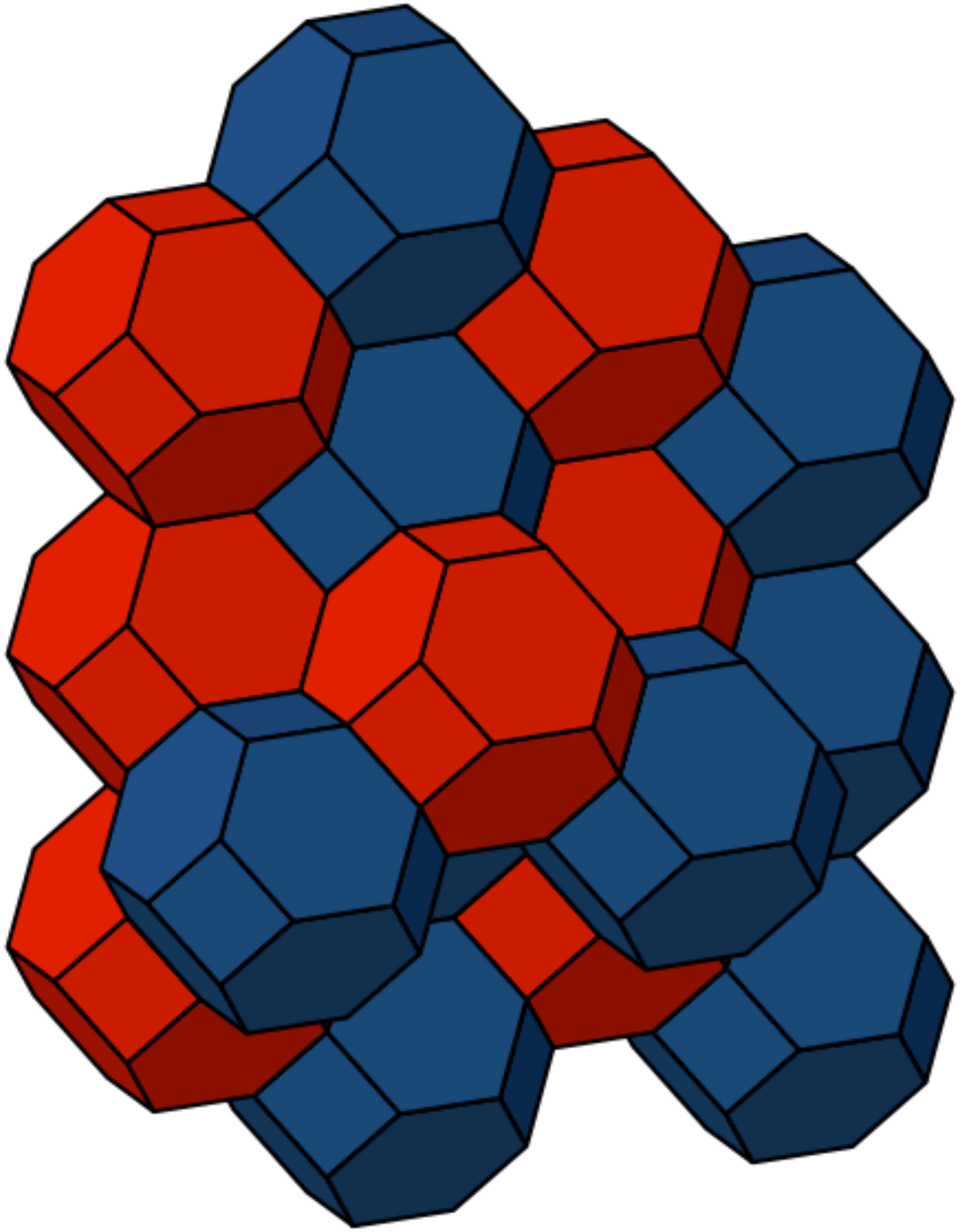
Honeycomb and Kelvin conjectures

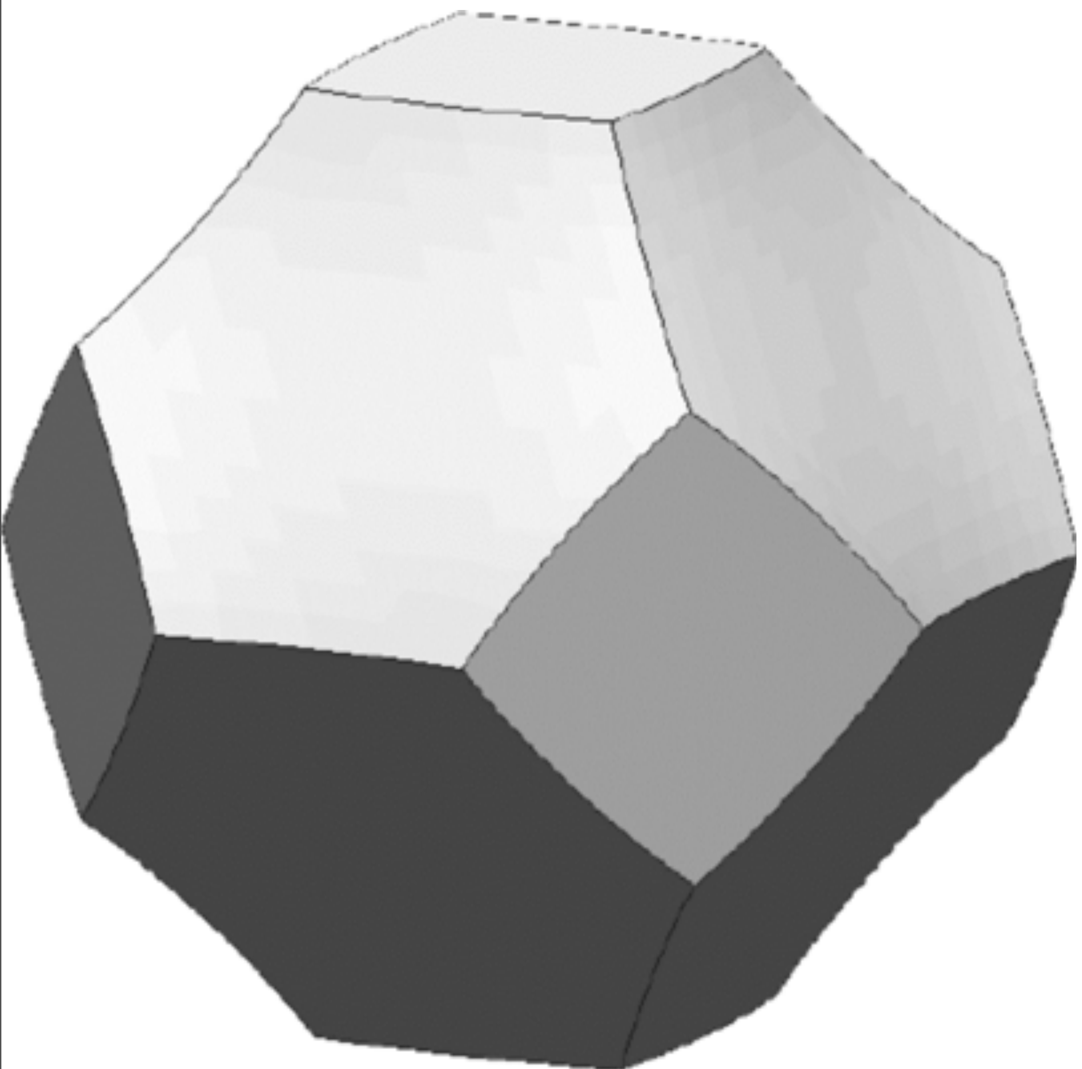
This honeycomb conjecture asserts that the hexagonal honeycomb is the optimal (perimeter-minimizing) partition of the plane into equal areas. The exact origins of this problem are not known, but the problem is very old. The first record of the conjecture dates back to 36 BC, from Marcus Terentius Varro, but is often attributed to Pappus of Alexandria (c.?290– c.?350).

> *Does not the chamber in the comb have six angles*
> *. . . The geometricians prove that this hexagon inscribed*
> *in a circular figure encloses the greatest amount of space.*
> *– Varro*

We can ask the corresponding problem in three dimensions. What partition of three-dimensional Euclidean space into equal volumes takes the least surface area? Kelvin made a conjecture in 1887, and it is now know as the Kelvin problem.





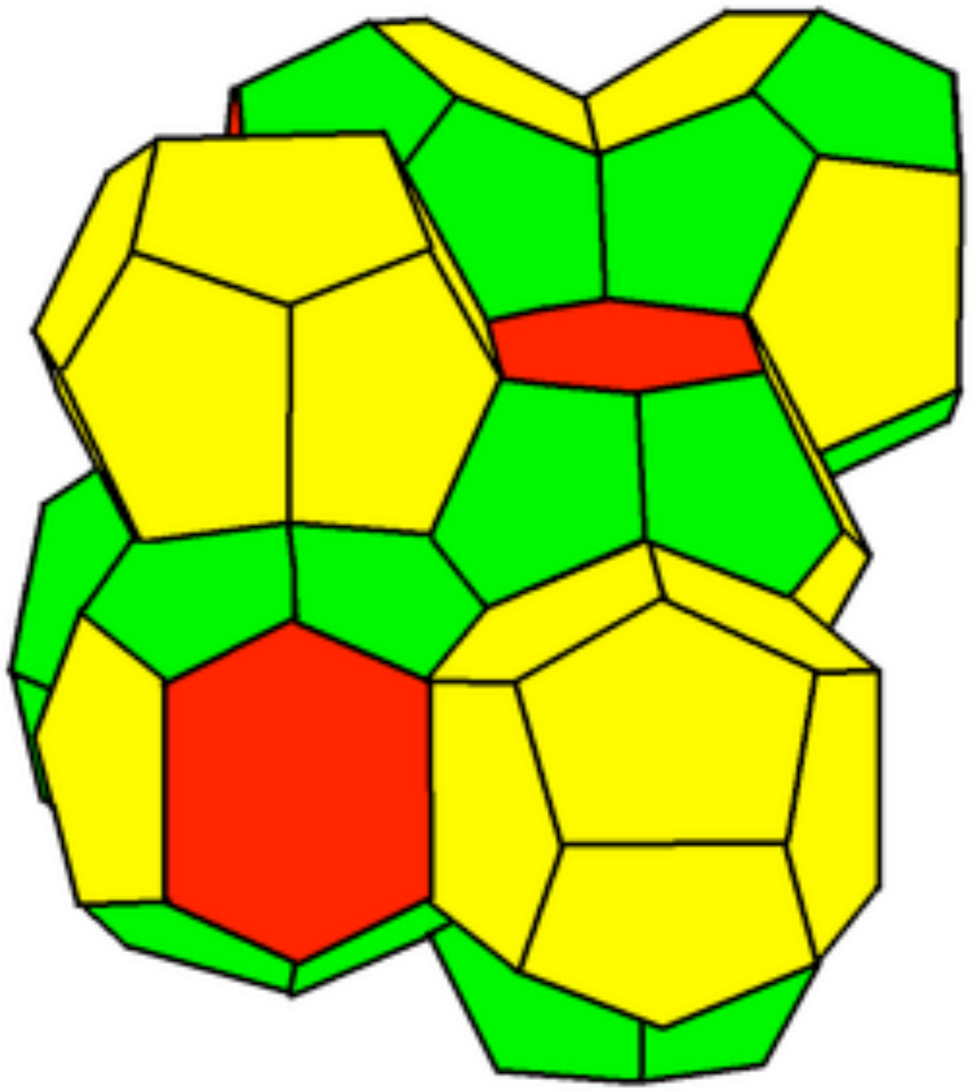
In fact, Kelvin (as a physicist) did not give a rigorous construction of his foam, and the existence of the Kelvin foam was itself just a matter of conjecture. Thus, there was never a Kelvin conjecture in the strict mathematical sense, only a *conjectural conjecture*.

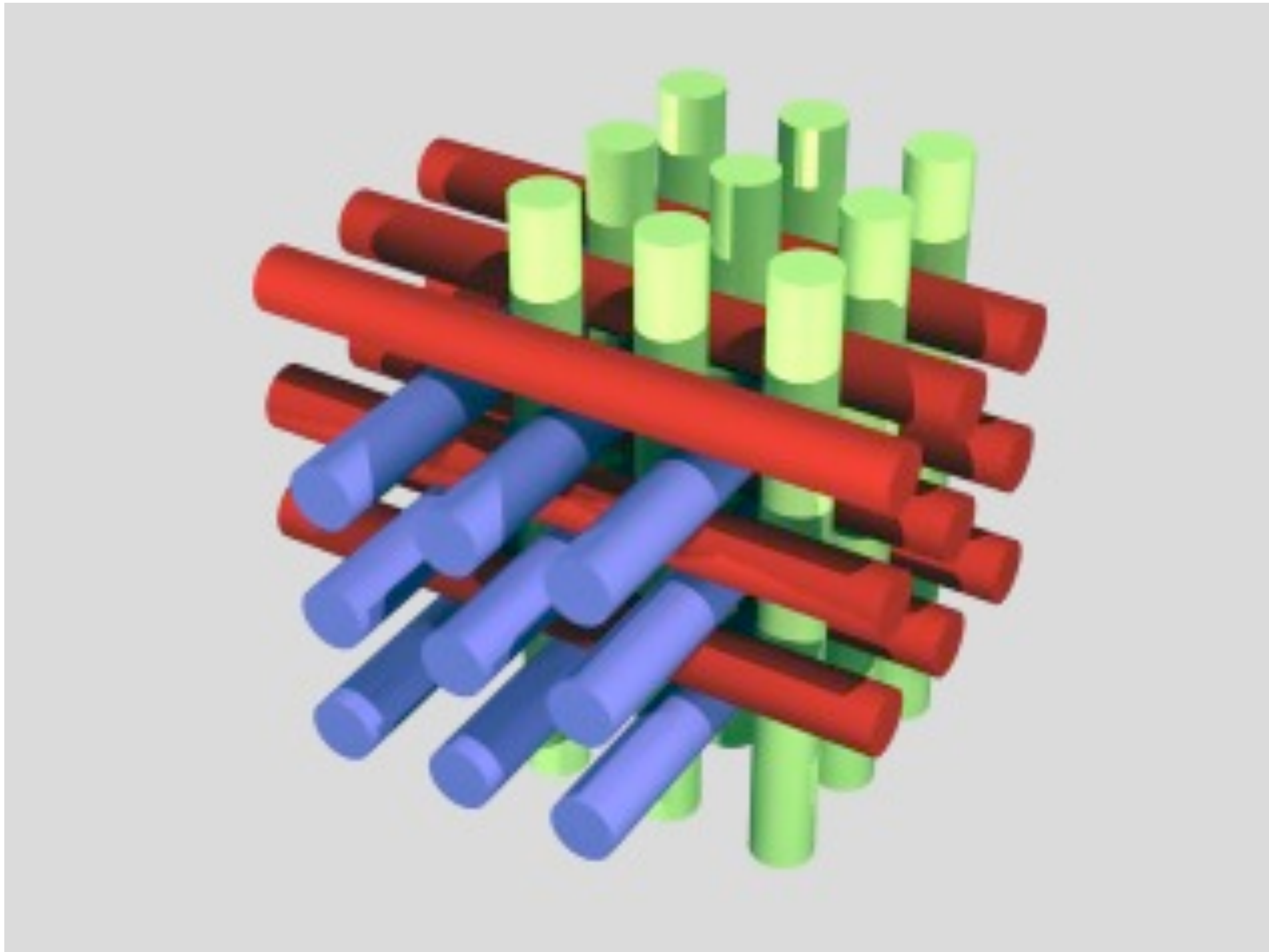
> *it is shown beautifully, and illustrated in great perfection, by making a skeleton model of 36 wire arcs for the 36 edges of the complete figure, and dipping it in soap solution to fill the faces with film, which is easily done for all the faces but one. The curvature of the hexagonal film on the two sides of the plane of its six long diagonals is beautifully shown by reflected light. – Kelvin*

The proof of Plateau's conditions did not come until Jean Taylor's proof in 1976. The proof of existence of the Kelvin foam was not announced until 1996, but one of the authors (F. Almgren, Kusner, Sullivan) passed away the next year, and the proof was never published.

(Speaking of big proof, Almgren's regularity theorem runs 955 pages.)

Weaire-Phelan foam (1993)





2008 Beijing Olympics, National Aquatics Center

Lesson: Do not assume that major conjectures are actually conjectures. Much can be learned about open problems by attempting to write them formally.

Everything is vague to a degree you do not realize till you have tried to make it precise. – Bertrand Russell

JOINT MATHEMATICS MEETINGS

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- **Thursday January 7, 2016, 8:00 a.m.-11:50 a.m.**
  AMS Special Session on Mathematical Information in the Digital Age of Science, III
  Room 603, Washington State Convention Center
  Organizers:
  Patrick Ion, University of Michigan, Ann Arbor pion@umich.edu
  Olaf Teschke, zbMATH, Berlin
  Stephen Watt, University of Western Ontario
  - 8:00 a.m.
    Formal Proof.

Formal Abstracts

On Digital Math Libraries

We should not compromise rigorous mathematical standards as we move from paper to computer. In fact, this is an opportunity to drastically improve standards. Many computer bugs are simply slips in logical and mathematical reasoning made by programmers and software designers.

- Mathematics influences the standards of scientific discourse, in the statistical sciences, in computer science, and throughout the sciences. If we promote sloppy platforms, the entire world will be worse off.
- Bugs in computer systems can lead to disaster: Intel Pentium FDIV bug, Ariane V explosion, . . .
- Bugs and design weaknesses in cryptographic software can be exploited by adversaries: Heartbleed, Logjam, Freak bug, . . .

A concrete proposal: mathematical FABSTRACTS (formal abstracts)

Given today's technology, it is not reasonable to ask for all proofs to be formalized. But with today's technology, it seems that it should be possible to create a formal abstract service that

- Gives a statement of the main theorem(s) of each published mathematical paper in a language that is both human and machine readable,
- Links each term in theorem statements to a precise definition of that term (again in human/machine readable form), and
- Grounds every statement and definition is the system in some foundational system for doing mathematics.

If we do not do proofs, there are various issues that come up.

- It is well-known in the formalization community that it is extremely difficult to get definitions correct until theorems are proved about the definitions.
- Mathematicians are notorious bad at giving the complete context needed for definitions.
- Example: one definition in the formal proof of the Kepler conjecture took nearly 40 revisions to get right.
- Various definitions are *specifications* that take a theorem asserting the existence of something, and that thing that exists is given a name. Definition building cannot take place without theorem proving.
- Building definitions requires non-obvious identifications: $\mathbb{Q}_p$ is the completion of $\mathbb{Q}_p$ with respect to the nonarchimedean metric, and it is the tensor product of $\mathbb{Q}$ and $\mathbb{Z}_p =$ the profinite limit of $\mathbb{Z}/\mathbb{Z}_p$.
- Example: TFAE. Define X to be any of the equivalent properties.

Solution: Do FORMAL-ABSTRACTS anyway.

Short term goals for FORMAL-ABSTRACTS

- Move beyond the subject areas of math that are traditionally covered by formalization projects.
- Automorphic Representation theory
- Clay Problem statements

Formalizing statements in Automorphic Representation Theory (a branch of number theory).

This will require a great deal of work just to state the theorems that are proved: algebraic geometry (schemes, motives, stacks, moduli spaces, and sheaves), measure theory and functional analysis, algebra (rings, modules, Galois theory, homological algebra, derived categories), category theory, complex analysis (L-functions and modular forms), class field theory (local and global), Lie theory and linear algebraic groups (Cartan classification and structure theory), representation theory (infinite dimensional, spectral theory), Shimura varieties, locally symmetric spaces, Hecke operators, cohomology (singular, deRham, intersection homology, l-adic), rigid geometry, perfectoid spaces,. . .

Clay Prize Problems

The Clay Problems (million-dollar problems)

- Birch-Swinnerton-Dyer: elliptic curves and L-functions
- Poincaré conjecture: 3-sphere, manifold, homotopy
- Hodge conjecture: deRham cohomology (differential forms), complex varieties
- Navier Stokes equations: easy statement PDE in 3-dimension.
- P versus NP: easy statement (Turing machine)
- Riemann hypothesis: Riemann zeta function is in HOL-Light. Analytic continuation?
- Yang-Mills and mass gap: it probably cannot be stated formally. The problem requires the winner to build a theory.

//physics.stackexchange.com/questions/339234/what-is-a-precise-mathematical-statement-of-the-yang-mills-and-

PHYSICS

QUESTIONS TAGS USERS

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Here's how it works:

Anybody can ask a question

Anybody can answer

What is a precise mathematical statement of the Yang-Mills and mass

8

5

I am a mathematician writing a statement of each of the Clay Millennium Prize problems in a formal proof assistant. For the other problems, it seems quite routine to write the conjectures formally, but I am having difficulty stating the problem on Yang Mills and the mass gap.

To me, it seems the Yang-Mills Clay problem is not a mathematical conjecture at all, but an under-specified request to develop a theory in which a certain theorem holds. As such, it is not capable of precise formulation. But a physicist I discussed this with believes that a formal mathematical conjecture should be possible.

I understand the classical Yang-Mills equation with gauge group G, as well as the Wightman axioms for QFT (roughly at the level of the IAS/QFT program), but I do not understand the requirements of the theory that link YM with Wightman QFT.

The official Clay problem from page 6 of Jaffe and Witten states the requirements (in extremely vague terms) as follows:

The official Clay problem from page 6 of Jaffe and Witten states the requirements (in extremely vague terms) as follows:

> "To establish existence of four-dimensional quantum gauge theory with gauge group G one should define a quantum field theory (in the above sense) with local quantum field operators in correspondence with the gauge-invariant local polynomials in the curvature F and its covariant derivatives [...]. Correlation functions of the quantum field operators should agree at short distances with the predictions of asymptotic freedom and perturbative renormalization theory, as described in textbooks. Those predictions include among other things the existence of a stress tensor and an operator product expansion, having prescribed local singularities predicted by asymptotic freedom."

A few phrases are somewhat clear to me like "gauge-invariant local polynomials...", but I do not see how to write much of this with mathematical precision. Can anyone help me out?

mathematical-physics gauge-theory yang-mills confinement wightman-fields

share cite improve this question

edited Jun 14 at 4:40
Qmechanic ♦
78.1k 11 126 451

asked Jun 13 at 21:59
Thales
41 1

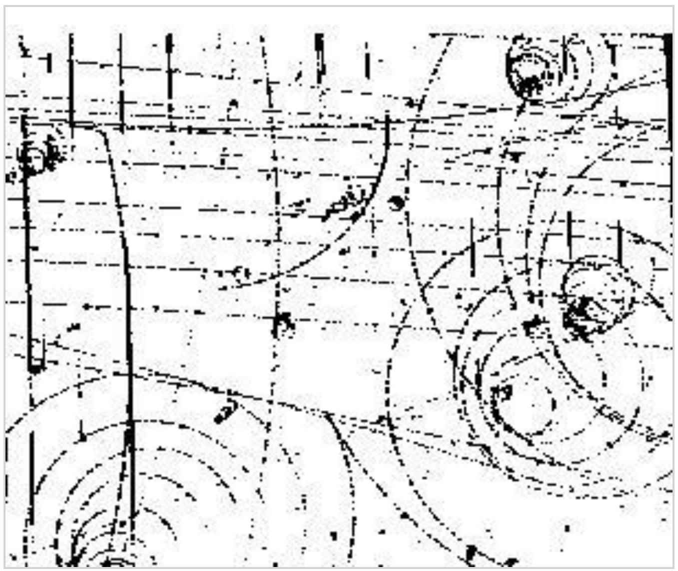
1 Related: physics.stackexchange.com/q/118825/2451 and links therein. – Qmechanic ♦ Jun 14 at 4:38

I think a good starting point could be F. Strocchi, Selected Topics on the General Properties of Quantum Field Theory , (World Scientific, Singapore, 1993). – Jon Jun 14 at 12:31

1 Possibly related on MathOverflow. – Keith McClary Jun 16 at 1:54

Jaffe and Witten's statement of the problem is probably the most precise you can get with our current state of knowledge. – SCFT Jun 18 at 4:59

Yang–Mills and Mass Gap



The laws of quantum physics stand to the world of elementary particles in the way that Newton's laws of classical mechanics stand to the macroscopic world. Almost half a century ago, Yang and Mills introduced a remarkable new framework to describe elementary particles using structures that also occur in geometry. Quantum Yang-Mills theory is now the foundation of most of elementary particle theory, and its predictions have been tested at many experimental laboratories, but its mathematical foundation is still unclear. The successful use of Yang-Mills theory to describe the strong interactions of elementary particles depends on a subtle quantum mechanical property called the "mass gap": the quantum particles have positive masses, even though the classical waves travel at the speed of light. This property has been discovered by physicists from experiment and confirmed by computer simulations, but it still has not been understood from a theoretical point of view. Progress in establishing the existence of the Yang-Mills theory and a mass gap will require the introduction of fundamental new ideas both in physics and in mathematics.

This problem is: Unsolved

Rules:

Rules for the Millennium Prizes

Related Documents:

Official Problem Description

Status of the Problem by Michael Douglas

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