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歐幾里德量子場論的變分方法

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(A variational approach to Euclidean QFT)

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▷ The Φ^4_3 measure ν is given by the formal prescription

$$\nu(\mathrm{d}\phi) = \frac{e^{-\lambda V(\phi)}}{\mathscr{Z}} \mu(\mathrm{d}\phi), \qquad V(\phi) = \int_\Lambda \phi(x)^4 \mathrm{d}x,$$

where μ is the Gaussian measure on $\mathscr{S}'(\Lambda)$ with covariance $(1-\Delta)^{-1}$, $\Lambda \subseteq \mathbb{R}^3$, $\lambda \geqslant 0$.

▷ The measure μ is only supported on distributions of regularity $(2-d)/2-\kappa$, therefore the potential V is not well defined $\Rightarrow$ need for renormalization.

▷ Regularization $\phi_T = \rho_T * \phi$ with $\rho_T \to \delta$ as $T \to \infty$ and introduction of *counterterms*

$$\nu_T(\mathrm{d}\phi) = \frac{e^{-\lambda V_T(\phi_T)}}{\mathscr{Z}_T} \mu(\mathrm{d}\phi), \qquad V_T(\phi) = \int_\Lambda (\phi^4 - a_T \phi^2 - b_T) \mathrm{d}x \geqslant -C_T > -\infty.$$

Problem: Control the limit $T \to \infty$ of the family $(\nu_T)_T$, describe the limiting object, prove the properties needed for applications to QFT (e.g. Osterwalder–Schrader axioms).

- ▷ *Constructive QFT*. ('70-'80) Glimm, Jaffe. Nelson. Segal. Guerra, Rosen, Simon...
- ▷ $(\Phi^4_3)_\Lambda$ Glimm ('69). Glimm, Jaffe. Feldman ('74), Y.M.Park ('75)
- ▷ $(\Phi^4_3)_{\mathbb{R}^3}$ Feldman, Osterwalder ('76). Magnen, Senéor ('76). Seiler, Simon ('76)
- ▷ *Other constructions*. Benfatto, Cassandro, Gallavotti, Nicolò, Olivieri, Presutti, Scacciatelli ('80) Brydges, Fröhlich, Sokal ('83) Battle, Federbush('83) Williamson ('87) Balaban ('83) Gawedzki, Kupiainen ('85) Watson ('89) Brydges, Dimock, Hurd ('95)
- ▷ *Stochastic quantisation (d=2)*. Jona-Lasinio, P.K.Mitter ('85) Borkar, Chari, S.K.Mitter ('88) Albeverio, Röckner ('91) Da Prato, Debussche ('03) Mourrat, Weber ('17) Röckner, R.Zhu, X.Zhu ('17)
- ▷ *Stochastic quantisation (d=3)*. Hairer ('14) Kupiainen ('16) Catellier, Chouk ('17) Mourrat, Weber ('17) Hairer, Mattingly ('18) R.Zhu, X.Zhu ('18) G, Hofmanova ('18)
- ▷ *Tightness via dynamics (d=3)*. Albeverio, Kusuoka ('18) G, Hofmanova ('18)

▷ As $T \to \infty$ fluctuations at different scales adds up independently into $(\phi_T)_T$.

▷ Wilson ('83) Polchinski ('84) Brydges, Kennedy ('87) Brydges, Dimock, Hurd ('95) Brydges, Slade, P.K.Mitter ('14)

▷ **HJB.** Formally the functional (effective potential)

$$U_t(\psi) := -\log \int e^{-\lambda V_T(\psi + \phi_T - \phi_t)} \mu(\mathrm{d}\phi), \qquad U_0(0) = -\log \mathscr{Z}_T,$$

is solution to an Hamilton–Jacobi–Bellman equation (flow equation)

$$\partial_t U_t(\psi) = -Q_t \left[\frac{\delta^2}{\delta\psi\delta\psi} U_t(\psi) + \frac{\delta U_t(\psi)}{\delta\psi} \frac{\delta U_t(\psi)}{\delta\psi} \right], \qquad U_T(\psi) = V_T(\psi).$$

This equation has to be studied in the space of functions over $\mathscr{S}'(\Lambda)$. Proper topology not very clear, diffusion is highly degenerate, not many (none?) results from the PDE point of view.

▷ We look for a stochastic control formulation of our problem which avoids the HJB equation.

▷ **Aim.** Present a new proof of existence of the limit $\nu_T \to \nu$.

▷ **Variational description.** Gibbs measures satisfy a variational principle, ν_T is the unique minimizer of the functional

$$G_T(\nu) = \lambda \int V_T(\phi)\mathrm{d}\nu(\phi) + H(\nu|\mu), \qquad G_T(\nu_T) = \inf_\nu G_T(\nu) = -\log \mathscr{Z}_T,$$

where $H(\nu|\mu) \geqslant 0$ is relative entropy.

The control of the limit $\nu_T \to \nu$ would follow from the Γ–convergence of the family of variational functionals $(G_T)_T$. Not clear how to obtain the needed estimates from the expression of G_T.

▷ If the probability space is generated by a Brownian motion $(B_t)_t$ the variational formula becomes more precise. $\mathbb{P}$ Wiener measure, X canonical process: if $\mathbb{Q} \ll \mathbb{P}$ then there exists $(u_s)_{s\geqslant 0}$ (Föllmer drift) such that

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^\infty u_s \mathrm{d}X_s - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right), \qquad H(\mathbb{Q}|\mathbb{P}) = \frac{1}{2}\mathbb{E}_{\mathbb{Q}}\left[\int_0^\infty |u_s|^2 \mathrm{d}s\right].$$

$\mathbb{P}$ Wiener measure, X canonical process.

Theorem. (Boué–Dupuis) *We have the variational representation*

$$-\log \mathbb{E}[e^{-F(X)}] = \inf_{u \in \mathbb{H}_a} \mathbb{E}_{\mathbb{P}}\left[F\left(X + \int_0^{\cdot} u_s \mathrm{d}s\right) + \frac{1}{2}\int_0^{\infty} |u_s|^2 \mathrm{d}s\right].$$

▷ Control problem (non–Markovian in general). Useful to get estimates and large deviations.

▷ Pathwise point of view: we shift the attention from the value function of optimization problem to the actual control needed to attain it. The problem becomes amenable to standard functional analysis techniques.

▷ The controlled process $X + \int_0^{\cdot} u_s \mathrm{d}s$ features explicitly the "free" part X and more regular drift part, similar to solutions to SDEs.

▷ Boué–Dupuis ('98), X. Zhang ('09), Lehec ('13), Üstünel ('14).

Let $F(X) \geqslant 0$ be Lipshitz, i.e.

$$|F(X+I(u))-F(X)| \leqslant L\|I(u)\|_{L^\infty([0,1])} \leqslant L\int_0^1 |u_s| \mathrm{d}s$$

Then

$$\log \mathbb{E}[e^{\lambda F(X)}] = \sup_u \mathbb{E}_{\mathbb{P}}\left[\lambda F(X+I(u)) - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right]$$

$$\leqslant \mathbb{E}_{\mathbb{P}}\left[\lambda F(X) + L\|I(u)\|_{L^\infty} - \frac{1}{2}\int_0^\infty |u_s|^2 \mathrm{d}s\right]$$

$$\leqslant \mathbb{E}_{\mathbb{P}}[\lambda F(X) + \underbrace{\frac{1}{2}\int_0^1 (2\lambda L|u_s| - |u_s|^2)\mathrm{d}s}_{\leqslant -\frac{1}{2}\lambda^2 L^2}] \leqslant \mathbb{E}_{\mathbb{P}}[\lambda F(X)] - \frac{1}{2}\lambda^2 L^2.$$

We conclude that F has Gaussian tails. Note that the only additional information needed is $\mathbb{E}_{\mathbb{P}}[|F(X)|] < +\infty$.

Note that L can be random, i.e. $L = L(X)$.

▷ Fix $\Lambda = \mathbb{T}^3$. Let X be a cylindrical Brownian motion on $L^2(\Lambda)$ and

$$Y_t = \int_0^t \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} \mathrm{d}X_s, \qquad \int_0^t \sigma_s(\mathrm{D})^2 \mathrm{d}s = \rho_t(\mathrm{D})^2$$

with $\mathrm{D} = |-\Delta|^{1/2}$, $\rho_t(\mathrm{D}) = \rho(\mathrm{D}/t)$ and $\rho \colon \mathbb{R}_+ \to \mathbb{R}_+$ smooth, compactly supported and with $\rho(0) = 1$. Then

$$\mathbb{E}_{\mathbb{P}}[Y_T(f) Y_S(g)] = \int_0^{T \wedge S} \left\langle \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} f, \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} g \right\rangle \mathrm{d}s = \left\langle f, \frac{\rho_{T \wedge S}(\mathrm{D})^2}{\langle \mathrm{D} \rangle^2} g \right\rangle,$$

so $Y_T \sim \rho_T * Y_\infty \sim \rho_T * \phi$ and $(Y_t)_t$ is a martingale.

Boué–Dupuis formula:

$$-\log \mathscr{Z}_T = -\log \mathbb{E}[e^{-\lambda V_T(Y_T(X))}] = \inf_{u \in \mathbb{H}_a} \mathbb{E}_{\mathbb{P}}\left[\lambda V_T(Y_T + Z_T) + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s\right]$$

with

$$Y\left(X + \int_0^\cdot u_s \mathrm{d}s\right) = Y_T + Z_T, \qquad Z_t = I_t(u) := \int_0^t \frac{\sigma_s(\mathrm{D})}{\langle \mathrm{D} \rangle} u_s \mathrm{d}s.$$

▷ **Regularity estimate.**

$$\sup_{0 \leqslant t \leqslant T} \|I_t(v)\|_{H^1}^2 \lesssim \int_0^T \|v_s\|_{L^2}^2 \mathrm{d}s.$$

So, at least heuristically,

$$\mathbb{E}_{\mathbb{P}}\left[\lambda V_T(Y_T + Z_T) + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s\right] \simeq \mathbb{E}_{\mathbb{P}}\left[\lambda V_T(Y_T + Z_T) + \frac{1}{2}\|\nabla Z_T\|^2\right]$$

▷ When $d=2$ we can choose the renomalization constants such that

$$\Theta_T(u) := \lambda V_T(Y_T + Z_T) + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s = \Psi_T(u) + \Phi_T(u)$$

$$\Psi_T(u) := \lambda \int_\Lambda [\![Y_T]\!]^4 + 4\lambda \int_\Lambda [\![Y_T^3]\!] Z_T + 6\lambda \int_\Lambda [\![Y_T^2]\!] Z_T^2 + 4\lambda \int_\Lambda [\![Y_T]\!] Z_T^3 + \Theta_T(u)$$

$$\Phi_T(u) := \underbrace{\lambda \int_\Lambda Z_T^4 + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s}_{\text{good terms}}$$

where $[\![Y_T^k]\!]$ are Wick polynomials of the (smooth) Gaussian field $(Y_T)_T$. In particular $T \mapsto [\![Y_T^k]\!]$ is a martingale.

▷ Standard estimates show that $[\![Y_T^k]\!] \in C([0,\infty], \mathscr{C}^{-\kappa}(\Lambda))$ almost surely with $L^p(\mathbb{P})$ norms for all $p \geqslant 1$ and $\kappa < 0$. Here $\mathscr{C}^\alpha(\Lambda) = B^\alpha_{\infty,\infty}(\Lambda)$ are Hölder–Besov spaces of regularity $\alpha \in \mathbb{R}$.

Now the game is to control the terms without sign with the good terms. Let $W_T = Y_T$.

$$\left|4\lambda \int_\Lambda [\![W_T^3]\!] Z_T\right| \leqslant 4\lambda \|[\![W_T^3]\!]\|_{H^{-1}} \|Z_T\|_{H^1} \leqslant C(\delta, d)\lambda^2 \|[\![W_T^3]\!]\|_{H^{-1}}^2 + \delta \int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s$$

$$\left|6\lambda \int_\Lambda [\![W_T^2]\!] Z_T^2\right| \leqslant \frac{C^2\lambda^3}{2\delta} \|[\![W_T^2]\!]\|_{W^{-\varepsilon,5}}^4 + \delta\left(\|Z_T\|_{W^{1,2}}^2 + \lambda \|Z_T\|_{L^4}^4\right)$$

$$\left|4\lambda \int_\Lambda W_T Z_T^3\right| \leqslant CE(\lambda) \|W_T\|_{W^{-1/2-\varepsilon,p}}^K + \delta\left(\|Z_T\|_{W^{1,2}}^2 + \lambda \|Z_T\|_{L^4}^4\right)$$

Therefore

$$-K_T + (1-\delta)\Phi_T(u) \leqslant \mathbb{E}[\Psi_T(u) + \Phi_T(u)] \leqslant K_T + (1+\delta)\Phi_T(u),$$

which implies

$$\sup_T |\log \mathscr{Z}_T| = \sup_T \left| \inf_{u \in \mathbb{H}_a} \mathbb{E}_{\mathbb{P}}[\Psi_T(u) + \Phi_T(u)] \right| \lesssim O(\lambda^2).$$

▷ In three dimensions W_∞ is more irregular and as a consequence we get uniform estimates for the Wick powers only in the following spaces

$$[\![W_T]\!] \in \mathscr{C}^{-1/2-\kappa}, [\![W_T^2]\!] \in \mathscr{C}^{-1-\kappa}, [\![W_T^3]\!] \in \mathscr{C}^{-3/2-\kappa}.$$

▷ As a consequence we cannot hope to control the term $\int_\Lambda [\![W_T^3]\!] Z_T$, and $\int_\Lambda [\![W_T^2]\!] Z_T^2$ as we did in two dimensions. Indeed we only have control of Z_T in H^1 and L^4.

▷ By perturbative considerations one expects further divergences (beyond Wick ordering) therefore the functional to minimize is now

$$\mathbb{E}\left[\lambda \int_\Lambda \mathbb{W}_T^3 Z_T + \frac{\lambda}{2} \int_\Lambda \mathbb{W}_T^2 Z_T^2 + 4\lambda \int_\Lambda W_T Z_T^3\right]$$

$$-\mathbb{E}\left[2\gamma_T \int_\Lambda W_T Z_T + \gamma_T \int_\Lambda Z_T^2\right] + \mathbb{E}\left[\lambda \int_\Lambda Z_T^4 + \frac{1}{2}\int_0^T \|u_s\|_{L^2}^2 \mathrm{d}s\right].$$

where we introduced the convenient notations: $\mathbb{W}_t^3 := 4[\![W_t^3]\!]$, $\mathbb{W}_t^2 := 12[\![W_t^2]\!]$.

▷ We aim to "complete the square" in order to eliminate the terms which we cannot control. So we control the system which a drift of the form

$$u_s = -\lambda J_s(\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s) + w_s$$

$$\dot{Z}_s = J_s u_s = -\lambda J_s^2(\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s) + \dot{K}_s$$

where w is a free control and $J_s = \langle \mathrm{D} \rangle^{-1} \sigma_s(\mathrm{D})$.

▷ *Paraproducts.* $fg = f \prec g + f \circ g + f \succ g$. (Bony, Meyer ('80))

▷ The cost of such a drift is

$$\frac{1}{2}\int_0^T \|u_s\|^2 \mathrm{d}s = \frac{\lambda^2}{2}\int_0^T \int_\Lambda (J_s(\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s))^2 \mathrm{d}s$$

$$-\lambda \int_0^T \int_\Lambda (\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s)\dot{Z}_s \mathrm{d}s + \frac{1}{2}\int_0^T \|w_s\|^2 \mathrm{d}s$$

▷ Integration by parts in the time variable allows to transform the mixed terms in this cost to

$$-\lambda \int_0^T \int_\Lambda (\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ Z_s) \dot{Z}_s \mathrm{d}s = -\lambda \int_\Lambda (\mathbb{W}_T^3 + \mathbb{W}_T^2 \succ Z_T) Z_T$$

$$+\lambda \int_0^T \int_\Lambda \left(\mathbb{W}_s^3 + \mathbb{W}_s^2 \succ \dot{Z}_s\right) Z_s \mathrm{d}s + \text{martingale}$$

which after some analysis will cancel the terms

$$\lambda \int_\Lambda (\mathbb{W}_T^3 Z_T + \mathbb{W}_T^2 Z_T^2)$$

modulo some nice remainder.

▷ The quadratic term generated by the new cost looks like (again after some integration by parts)

$$\frac{\lambda^2}{2}\int_0^T\int_\Lambda (J_s(\mathbb{W}_s^3+\mathbb{W}_s^2\succ Z_s))^2\mathrm{d}s=\frac{\lambda^2}{2}\int_0^T\int_\Lambda (J_s(\mathbb{W}_s^3))^2\mathrm{d}s$$

$$+\frac{\lambda^2}{2}\int_0^T\int_\Lambda [(J_s(\mathbb{W}_s^2\succ Z_s))^2-2\dot{\gamma}_s Z_s^2]\mathrm{d}s$$

$$+\lambda^2\int_0^T\int_\Lambda [(J_s(\mathbb{W}_s^3))(J_s(\mathbb{W}_s^2\succ Z_s))-2\dot{\gamma}_s W_s Z_s]\mathrm{d}s+\lambda^2\int_0^T\int_\Lambda \dot{\gamma}_s[(Z_s)^2+2W_s Z_s]\mathrm{d}s$$

where we have introduced an abitrary function $(\gamma_s)_s$. In this expression now the first term is divergent but independend of the control, the two middle terms can be shown to be finite provided the counterterm γ is chosen appropriately and finally, the last term is compensated by

$$2\gamma_T\int_\Lambda W_T Z_T+\gamma_T\int_\Lambda Z_T^2.$$

Let us see how does it work for

$$A = \frac{\lambda^2}{2} \int_0^T \int_\Lambda [(J_s(\mathbb{W}_s^2 \succ Z_s))^2 - 2\dot{\gamma}_s Z_s^2] \mathrm{d}s.$$

▷ *Commutator lemma.* $J_s \mathbb{W}_s^2 \in \mathscr{C}^{-\kappa}$ and $Z_s \in H^{1/2-\kappa}$

$$\int_\Lambda (J_s(\mathbb{W}_s^2 \succ Z_s))^2 = \int_\Lambda (J_s(\mathbb{W}_s^2 \succ Z_s)) \circ (J_s(\mathbb{W}_s^2 \succ Z_s))$$

$$\simeq \int_\Lambda (J_s \mathbb{W}_s^2) \circ (J_s \mathbb{W}_s^2) Z_s^2 + \int_\Lambda \underbrace{C(J_s \mathbb{W}_s^2, J_s \mathbb{W}_s^2, Z_s)}_{\in B_{1,1}^{0+}}$$

Therefore

$$A = \frac{\lambda^2}{2} \int_0^T \int_\Lambda \underbrace{[(J_s \mathbb{W}_s^2) \circ (J_s \mathbb{W}_s^2) - 2\dot{\gamma}_s]}_{\mathbb{W}^{2\diamond 2} \in \mathscr{C}^{-\kappa}} Z_s^2 \mathrm{d}s$$

Similarly

$$\mathbb{W}_s^{2\diamond 3} := (J_s \mathbb{W}_s^3) \circ (J_s \mathbb{W}_s^2) - 2\dot{\gamma}_s W_s \in \mathscr{C}^{-1/2-\kappa}$$

$$\mathbb{W}_T := (W_T, \mathbb{W}_T^2, \mathbb{W}_T^3, \mathbb{W}^{2\diamond 2}, \mathbb{W}_s^{2\diamond 3}) \in \mathfrak{W} = \mathscr{C}^{-1/2-\kappa} \times \mathscr{C}^{-1-\kappa} \times \mathscr{C}^{-3/2-\kappa} \times \mathscr{C}^{-\kappa} \times \mathscr{C}^{-1/2-\kappa}$$

▷ We have shown that

$$\begin{aligned} -\log \mathscr{Z}_T(\lambda) &= \inf_{u \in \mathbb{H}_a} \mathbb{E}\left[\lambda V_T(Y_T + I_T(u)) + \frac{1}{2}\int_0^\infty \|u_s\|_{L^2}^2 \mathrm{d}s\right] \\ &= \inf_{l \in \mathbb{H}_a} \mathbb{E}\left[E_T(Z(l), K(l)) + \lambda \|Z_T(l)\|_{L^4}^4 + \frac{1}{2}\int_0^\infty \|l_s\|_{L^2}^2 \mathrm{d}s\right] \\ &=: \inf_{l \in \mathbb{H}_a} \tilde{F}_T(l) \end{aligned}$$

where $Z = Z(l) \in H^{1/2-\varepsilon}$ and $K = K(l) \in H^{1-\varepsilon}$ solve the integral equations

$$Z_t(l) = -\lambda \int_0^t J_s^2 \mathbb{W}_s^3 \mathrm{d}s + K_t(l), \qquad K_t(l) = -\lambda \int_0^t J_s^2 (\mathbb{W}_s^2 \succ Z_s(l)) \mathrm{d}s + \int_0^t J_s l_s \mathrm{d}s.$$

▷ Estimates of the form

$$|E_T(Z(l), K(l))| \leqslant C \|\mathscr{W}_T\|_S^K + \delta \|Z_T(l)\|_{L^4}^4 + \delta \|K(l)\|_{H^{1-\varepsilon}}^2.$$

Variational setting. (X,l) canonical variables on $C([0,\infty],\mathfrak{W})\times L^2_w([0,\infty)\times\Lambda)$

$$\mathscr{X} := \{\mu\in P(C([0,\infty],S)\times L^2_w([0,\infty)\times\Lambda))\,|\,\mu=\mathrm{Law}_{\mathbb{P}}(W,u)\text{ for some }u\in\mathbb{H}_a\}.$$

▷ Then

$$-\log\mathscr{Z}_T(\lambda)=\inf_{\mu\in\mathscr{X}}F_T(\mu)=\inf_{\mu\in\bar{\mathscr{X}}}F_T(\mu)$$

where, for $T\in[0,\infty]$,

$$F_T(\mu):=\mathbb{E}_\mu\left[E_T(Z(l),K(l))+\lambda\|Z_T(l)\|^4_{L^4(\Lambda)}+\frac{1}{2}\int_0^\infty\|l_s\|^2_{L^2}\mathrm{d}s\right].$$

▷ The choice of $\mathscr{X}$ is dictated by the fact that the family $(F_T)_T$ is now equicoercive, namely that there exists a compact $\mathscr{K}\subseteq\mathscr{X}$ such that

$$\inf_{x\in\mathscr{K}}F_T(x)=\inf_{x\in\mathscr{X}}F_T(x),\qquad\text{for all }T.$$

▷ Finally using the continuity of the map E and the lower semicontinuity of the L^4 and entropy terms we establish

$$\Gamma\text{-}\lim_{T\to\infty} F_T = F_\infty.$$

Namely that

- For every sequence $\mu^T \to \mu$ in $\bar{\mathscr{X}}$:

$$F_\infty(\mu) \leqslant \liminf_T F_T(\mu^T),$$

- For every $\mu \in \bar{\mathscr{X}}$ there exists a sequence $\mu^T \to \mu$ in $\bar{\mathscr{X}}$ such that

$$F_\infty(\mu) \geqslant \limsup_T F_T(\mu^T).$$

▷ A consequence of Γ–convergence is the convergence of minima:

$$\lim_{T\to\infty} (-\log \mathscr{Z}_T) = \lim_{T\to\infty} \inf_{\bar{\mathscr{X}}} F_T = \min_{\bar{\mathscr{X}}} F_\infty.$$

We obtain *explicit* variational formula for the limiting functional

$$-\log \mathscr{Z}_\infty(f) = \inf_{l\in\mathbb{H}_a} \mathbb{E}\left[-\int_\Lambda f Z_\infty(l) + E_\infty(Z(l), K(l)) + \lambda \|Z_\infty(l)\|^4_{L^4(\Lambda)} + \frac{1}{2}\int_0^\infty \|l_s\|^2_{L^2} \mathrm{d}s\right]$$

defined for all $f \in \mathscr{S}(\Lambda)$ with

$$\mathscr{Z}_\infty(f) = \lim_T \mathscr{Z}_T(f), \qquad \mathscr{Z}_T(f) = \mathscr{Z}_T \mathbb{E}_\nu[e^{\int_\Lambda f\phi_T}] = \int e^{\int_\Lambda f\phi_T - \lambda V_T(\phi_T)} \mu(\mathrm{d}\phi).$$

▷ The interest of this formula lies in the fact that the Φ^4_3 measure is not absolutely continuous wrt. the Gaussian free field, so an explicit description was lacking.

▷ The variational formula seems a promising way to extract informations from this measure. E.g. large deviations, weak universality, pathwise properties, etc...

$$E_\infty(Z(l),K(l)) = E_\infty(Z,K) = \sum_{i=1}^{6} \Upsilon_\infty^{(i)}$$

with

$$\begin{aligned}
\Upsilon_\infty^{(1)} &:= \frac{\lambda}{2}\kappa^{(2)}(\mathbb{W}_\infty^2, K_\infty, K_\infty) + \frac{\lambda}{2}\fint (\mathbb{W}_\infty^2 \prec K_\infty) K_\infty - \lambda^2 \fint (\mathbb{W}_\infty^2 \prec \mathbb{W}_\infty^{[3]}) K_\infty \\
\Upsilon_\infty^{(2)} &= 0 \\
\Upsilon_\infty^{(3)} &:= \lambda \int_0^\infty \fint (\mathbb{W}_t^2 \succ \dot{Z}_t^\flat) K_t \mathrm{d}t \\
\Upsilon_\infty^{(4)} &:= 4\lambda \fint \mathbb{W}_\infty K_\infty^3 + 12\lambda^2 \fint (\mathbb{W}_\infty \mathbb{W}_\infty^{[3]}) K_\infty^2 + 12\lambda^3 \fint \mathbb{W}_\infty (\mathbb{W}_\infty^{[3]})^2 K_\infty \\
\Upsilon_\infty^{(5)} &:= -2\lambda^2 \int_0^\infty \fint \gamma_t Z_t^\flat \dot{Z}_t^\flat \mathrm{d}t \\
\Upsilon_\infty^{(6)} &:= -\lambda^2 \fint \mathbb{W}_\infty^{2\diamond[3]} K_\infty - \lambda^2 \int_0^T \fint \mathbb{W}_t^{2\diamond 2} (Z_t^\flat)^2 \mathrm{d}t + \frac{\lambda^2}{2} \int_0^\infty \kappa_t^{(1)} (\mathbb{W}_t^2, Z_t^\flat, Z_t^\flat)
\end{aligned}$$

and

$$|\gamma_t| + \langle t \rangle |\dot{\gamma}_t| \lesssim \lambda^2 \log\langle t \rangle.$$

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