Bessel S(P)DEs: a story of renormalisation

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Motivation: model the random evolution of a continuous interface over an obstacle.

At equilibrium, such a system can be modeled by a *Bessel bridge*. The dynamics can be modeled by a *Bessel SPDE*. The corresponding equations involve a renormalisation of local times.

Squared Bessel processes

Let $\delta \geq 0$, $y \geq 0$, and $(B_t)_{t \geq 0}$ a BM.

By Yamada-Watanabe's Theorem, there exists a unique (strong) solution $(Y_t)_{t \geq 0}$ of

$$Y_t = y + \int_0^t 2\sqrt{|Y_s|}\,\mathrm{d}B_s + \delta\, t,$$

and moreover $Y \geq 0$ so that $|Y| = Y$.

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$(Y_t)_{t\geq 0}$ is called a Squared Bessel Process.
Many quantities related to these processes can be computed explicitly (see Pitman-Yor).

Bessel processes

We set $x := \sqrt{y}$, and define

$$X_t := \sqrt{Y_t}, \quad t \geq 0.$$

$(X_t)_{t\geq 0}$ is a continuous Markov process on $\mathbb{R}_+$, called δ-dimensional Bessel process started from x.

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Example (The values $\delta = 1$ and $\delta = 3$)

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Example (The values $\delta = 1$ and $\delta = 3$)

If B is a standard Brownian motion, then $X_t := |B_t|$ is a 1-dimensional Bessel process.

On the other hand, the conditional law:

$$\mathcal{L}(B_t, t \in [0,1] \,|\, B \geq 0)$$

is equivalent to the law of a 3-dimensional Bessel process on $[0, 1]$.

Diffusion local times of a Bessel process

Question: Given $b \geq 0$, how much time does a δ-dimensional Bessel process $(X_t)_{t \geq 0}$ spend at the level b ?

Proposition

There exists a continuous process $(\ell_t^b)_{b \geq 0, t \geq 0}$ *such that, almost-surely, for all* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *Borel and* $t \geq 0$ *:*

$$\int_0^t f(X_s)\,\mathrm{d}s = \int_0^\infty f(b)\,\ell_t^b\,b^{\delta-1}\,\mathrm{d}b.$$

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What can we say about $(\ell_t^0)_{t \geq 0}$ or, equivalently, about the contact set $Z(X) := \{t \geq 0, X_t = 0\}$?

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Proposition

- *If $\delta \geq 2$, then $Z(X) = \emptyset$ a.s.*
- *If $\delta < 2$, then $Z(X)$ is infinite a.s.*

Bessel SDE

What equation does a δ-dimensional Bessel process X satisfy?

For $\delta > 1$ we have:

$$X_t = x + \frac{\delta - 1}{2} \int_0^t \frac{1}{X_s} \, \mathrm{d}s + B_t.$$

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Finally, for $\delta \in (0, 1)$:

$$X_t = x + \frac{\delta - 1}{2} \int_0^\infty \frac{1}{b} (\ell_t^b - \ell_t^0) b^{\delta - 1} \, \mathrm{d}b + B_t,$$

where we see a renormalisation of local times $\ell_t^b - \ell_t^0$ appearing.

Definition

Let $(X_t)_{t\geq 0}$ be a δ-dimensional Bessel process started from 0. Then the probability law:

$$\mathcal{L}(X_t,\ t \in [0,1]\,|X_1 = 0)$$

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In a series of articles of the early 2000s, L. Zambotti introduced a family of parabolic SPDEs with properties analogous to Bessel processes. In particular, they admit Bessel bridges of dimension $\delta \geq 3$ as reversible measure.

In 1992, Nualart and Pardoux introduced and solved the stochastic heat equation with reflection on $\mathbb{R}_+ \times [0,1]$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x^2} + \xi + \eta, \\ \\ u \geq 0, \quad \mathrm{d}\eta \geq 0, \quad \int_{\mathbb{R}_+ \times [0,1]} u \,\mathrm{d}\eta = 0. \end{cases} \qquad (B_3)$$

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Let $\delta > 3$ and $c(\delta) := \frac{(\delta-1)(\delta-3)}{8} > 0$. Lorenzo considered the stochastic heat equation with repulsion from 0:

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{c(\delta)}{u^3} + \xi, \qquad (B_\delta)$$

where $u \geq 0$. This SPDE has the law of a δ-dimensional Bessel bridge as reversible measure.

The above SPDEs arise as scaling limits of various discrete interface models :

- $\nabla\phi$ interface models (Funaki-Olla, 2000 and Zambotti, 2004)
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- diffusive scaling ($1 - 2 - 4$)
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What about $\delta < 3$? This question has been open since 2001, and is particularly relevant for $\delta = 1$, in view of scaling limits of dynamical critical pinning models.

Lorenzo and I have recently proved integration by parts formulae on the laws of Bessel bridges of dimension $\delta < 3$. These formulae suggest that, when $\delta < 3$, the SPDE should be of the form

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial t}\mathrm{drift} + \xi$$

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More precisely, for all $x \in (0,1)$, there should exist a process $(\ell^b_{t,x})_{b,t\geq 0}$, such that:

$$\forall f : \mathbb{R}_+ \to \mathbb{R}_+, \quad \int_0^t f\,(u(s,x))\,ds = \int_0^{+\infty} f(b)\,\ell^b_{t,x}\,b^{\delta-1}\,\mathrm{d}b.$$

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Moreover, for all $x \in (0,1)$ and $t > 0$, we should have:

$$\left.\frac{\partial}{\partial b}\ell^b_{t,x}\right|_{b=0} = 0.$$

Then, for $\delta \in (1, 3)$, the drift would be formally given by :

$$c(\delta) \int_0^{+\infty} \frac{1}{b^3} \left(\ell_t^b(x) - \ell_t^0(x) \right) b^{\delta - 1} \, \mathrm{d}b$$

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$$c(\delta) \int_0^{+\infty} \frac{1}{b^3} \left(\ell_t^b(x) - \ell_t^0(x) - \frac{b^2}{2} \partial_u^2 \ell_t^u(x) \bigg|_{u=0} \right) b^{\delta-1} \, \mathrm{d}b.$$

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Finally, for $\delta = 1$, the drift would be formally given by:

$$-\frac{1}{8} \partial_u^2 \ell_t^u(x) \Big|_{u=0},$$

so that we could write the following SPDE for $\delta = 1$:

$$\partial_t u = \frac{1}{2} \partial_x^2 u - \frac{1}{8} \partial_t \partial_u^2 \ell_t^u(x) \Big|_{u=0} + \xi, \qquad (B_1)$$

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describing the scaling limit of dynamical critical pinning models.

Comments and open questions

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The above SPDEs lie at the crossroads of major open questions :

- local times for SPDEs ?
- pathwise well-posedness ?
- Strong Feller property ?
- the associated Dirichlet forms for $\delta \neq 1, 2$?

What is the relation between all the SPDEs above ?

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One can rewrite all the SPDEs above in a unified way, using a family of Schwartz distributions $(\mu_\alpha)_{\alpha\in\mathbb{R}}$ acting as fractional derivatives.

For all $\alpha > 0$, we consider the following measure on $\mathbb{R}_+$:

$$\mu_\alpha(\mathrm{d}x) := \frac{1}{\Gamma(\alpha)} x^{\alpha-1}\,\mathrm{d}x.$$

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For $\alpha = -k$ with $k \in \mathbb{N}$, we define the Schwartz distribution:

$$\langle \mu_\alpha, \varphi \rangle := (-1)^k \varphi^{(k)}(0), \quad \varphi \in C_c(\mathbb{R}_+).$$

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Finally, if $-k-1 < \alpha < -k$ with $k \in \mathbb{N}$, we set :

$$\langle \mu_\alpha, \varphi \rangle := \frac{1}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \left(\varphi(x) - \sum_{j=0}^{k} \frac{(-1)^j}{j!} \varphi^{(j)}(0) \right) \mathrm{d}x, \quad \varphi \in C_c(\mathbb{R}_+).$$

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The family of distributions $(\mu_\alpha)_{\alpha \in \mathbb{R}}$ has numerous nice properties, and is very similar to Caputo's notion of fractional derivative (see also Ramanujan's Master Theorem).

A unified fomula

For all $\delta > 0$, the δ-Bessel SDE can be rewritten:

$$X_t = X_0 + \frac{\Gamma(\delta)}{2} \langle \mu_{\delta-1}(\,\mathrm{d}b), \ell_t^b \rangle + B_t.$$

On the other hand, for all $\delta > 0$, the δ- Bessel SPDE can be rewritten:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\Gamma(\delta)}{8(\delta - 2)} \langle \mu_{\delta-3}(\,\mathrm{d}b), \ell_{t,x}^b \rangle + \xi.$$

A distinction result

Let $u(t,x)$ be a stationary solution to the stochastic heat equation on $\mathbb{R}_+ \times [0,1]$, i.e.:

$$\partial_t u = \frac{1}{2}\partial_x^2 u + \xi,$$

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$|u| = (|u_t|)_{t\geq 0}$ is a C_0- valued stochastic process. It solves an SPDE, which is an instance of an infinite-dimensional Tanaka formula (Zambotti, 2006).

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Theorem

$|u|$ *does not have the same law as a stationary solution to the Bessel-like SPDE for* $\delta = 1$.