Random walks and fractional Euler-Poisson-Darboux equation

Shishkina E.L.[1,2]
ilina_dico@mail.ru

[1]Voronezh State University,
[2] Belgorod State National Research University (BelGU)

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Introduction

The classical Euler–Poisson–Darboux (EPD) equation is defined by

$$\frac{\partial^2 u}{\partial t^2}+\frac{\gamma}{t}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}, \quad u=u(x,t), \quad x\in\mathbb{R}, \quad t>0, \quad \gamma\in\mathbb{R}. \quad (1)$$

The operator acting by variable t in (1) is the Bessel operator

$$(B_\gamma)_t=\frac{\partial^2}{\partial t^2}+\frac{\gamma}{t}\frac{\partial}{\partial t}.$$

The equation (1) is considered as a model of random flights in

- Orsingher E. A planar random motion governed by the two-dimensional telegraph equation. J. Appl. Probab. 1986;23:385–397.
- Orsingher E. Probability law, flow function, maximum distribution of wave-governed random motions, and their connections with Kirchhoff's laws. Stochastic Process Appl. 1990;34:49–66.

Introduction

The first contribution in this area goes back to Sydney Goldstein

- Goldstein S. On diffusion by discontinuous movements and thetelegraph equation. Quart. J. Mech. Appl. Math. 1951;4:129–156.

He considered the simplest random walk on the real line, in which a particle placed at the origin at time 0 moves with two finite speeds $\pm\lambda$ changing its current speed in accordance with the simplest Poisson process with a constant parameter μ. He discovered that the distribution of position particles x during t is a solution to a telegraph equation of the form

$$\frac{\partial^2 u}{\partial t^2} + 2\mu \frac{\partial u}{\partial t} = \lambda^2 \frac{\partial^2 u}{\partial x^2}.$$

Introduction

This model was then examined in detail by Mark Katz and Enzo Orsinger

- Katz M. A stochastic model related to the telegrapher's equation. Rocky Mountain J. Math. 1974;4:497–509.
- Orsingher E. Hyperbolic equations arising in random models. Stochastic Process Appl. 1985;21:93–106.

Introduction

Natural generalizations to the case of the Poisson process with the intensity function $\lambda = \lambda(t) \in C^1(\mathbb{R})$ and to the multidimensional case were examined in

- De Gregorio A.& Orsingher E. Random flights connecting porous medium and Euler-Poisson-Darboux equations. J. Math. Phys. 2020;61(4):1–18.
- Garra R. & Orsingher E. Random flights related to the Euler-Poisson-Darboux equation. Markov processes and related fields. 2016;22:87–110.
- Iacus S. Statistical analysis of the inhomogeneous telegrapher's process. Statistics & Probability Letters. 2001;55:83–88.

Introduction

Models of random walks with fractional derivatives were considered in

- Metzler R. & Klafter J. The random walk's guide to anomalous diffusion: A fractional dynamics approach. Physics Report. 2000;339:1–77.
- Gorenflo RR., Vivoli A. & Mainardi F. Discrete and continuous random walk models for space-time fractional diffusion. Nonlinear Dynamics. 2004;38:101–116.

Introduction

In

- De Gregorio A.& Orsingher E. Flying randomly in R^d with Dirichlet displacements. Stoch. Process. Appl. 2012;122(2):676–713.

it was shown that the Euler–Poisson–Darboux equation of the form

$$\frac{\partial^2 u}{\partial t^2} + \frac{\gamma}{t}\frac{\partial u}{\partial t} = \lambda^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad a > 0, \quad t > 0, \quad x \in \mathbb{R} \tag{2}$$

defines the probabilistic law of random walk on $\mathbb{R}$. The explicit distribution $u(x,t)$ of the position of arbitrarily moving particles is obtained by solving the initial problems for the Euler–Poisson–Darboux equation (2).

Introduction

In paper

- Garra R. & Orsingher E. Random flights related to the Euler-Poisson-Darboux equation. Markov processes and related fields. 2016;22:87–110.

fractional diffusion-wave equation

$$\left(\frac{\partial^2 u}{\partial t^2} + \frac{\gamma}{t}\frac{\partial u}{\partial t}\right)^{\alpha} u = \lambda^2 \frac{\partial^2 u}{\partial x^2}, \tag{3}$$

$$u = u(x,t), \qquad x \in \mathbb{R}, \qquad t > 0, \qquad 0 < \alpha < 1$$

was obtained as a model of random walk.

Introduction

Model

$$\left(\frac{\partial^2 u}{\partial t^2} + \frac{\gamma}{t}\frac{\partial u}{\partial t}\right)^{\alpha} u = \lambda^2 \frac{\partial^2 u}{\partial x^2},$$

means that for $\alpha \in (0, 1/2)$ the particle moves on average more slowly than when considering the model (2) which is corresponds to $\alpha = 1/2$. For $\alpha \in (1/2, 1)$ the particle moves faster on average. In this talk using operational method we solve fractional Euler–Poisson–Darboux equation of the form (3) with additional conditions for $0 < \alpha \leq 1/2$.

Fractional Euler–Poisson–Darboux equation

A. N. Gerasimov in

- Gerasimov AN. A generalization of linear laws of deformation and its application to problems of internal friction. Akad. Nauk SSSR, Prikl. Mat. Mekh. 1948;12:251–259

derived and solved fractional-order partial differential equation

$$\frac{\partial^{2\beta} u}{\partial t^{2\beta}} = D\frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad x \in \mathbb{R}, \quad t > 0, \quad 0 < \beta \qquad (4)$$

for viscoelasticity problems. The Cauchy problem for the equation (4) was considered by F. Mainardi in

- Mainardi F. The Fundamental Solutions for the Fractional Diffusion-Wave Equation. Appl. Math. Lett. 1996;9(6):23–28.

Fractional Euler–Poisson–Darboux equation

Also such kind of equations were considered by F. Mainardi and R. Gorenflo in

- Gorenflo R. & Mainardi F. Fractional calculus and stable probability distributions. Arch. Mech. (Basel). 1998;50(3):377–388,

by I. Podlubny in

- Podlubny I. Fractional Differential Equations. Academic Press: San Diego; 1999,

by A. A. Kilbas, H. M. Srivastava, J. J. Trujillo in

- Kilbas AA., Srivastava HM. & Trujillo JJ. Theory and applications of fractional differential equations. Elsevier: Amsterdam; 2006

and others.

Fractional Euler–Poisson–Darboux equation

Let consider first the simplest one-dimensional case when $u = u(x,t)$, $x \in \mathbb{R}$, $t \geq 0$,

$$(\mathcal{B}^{\alpha}_{\gamma,0+})_t u(x,t) = \lambda^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leq \alpha < 1/2, \qquad \lambda > 0, \tag{5}$$

with the Cauchy condition

$$u(x,0) = f(x). \tag{6}$$

Fractional Euler–Poisson–Darboux equation

Theorem

Let $0 < \alpha \leq 1/2$, $\lambda > 0$ then the solution to the problem (5)–(6) *is*

$$u(x,t) = \int\limits_{-\infty}^{\infty} G_\gamma^\alpha(x-\xi,t) f(\xi) d\xi, \tag{7}$$

where

$$G_\gamma^\alpha(x,t) =$$

$$= \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\lambda\sqrt{\pi}2^{1-\gamma}} t^{-\alpha} \mathbf{H}_{1,3}^{2,0}\left[\frac{|x|}{\lambda} t^{-\alpha} \middle| \begin{array}{c} \left(1-\frac{\alpha}{2},\frac{\alpha}{2}\right) \\ \left(1-\frac{\alpha-\gamma}{2},\frac{\alpha}{2}\right),(0,1),(\alpha-\gamma,-\alpha) \end{array}\right].$$

provided that the integral in the right-hand side of (7) *is convergent.*

Fractional Euler–Poisson–Darboux equation

For integers m, n, p, q such that $0 \leq m \leq q$; $0 \leq n \leq p$, $a_i, b_j \in \mathbb{C}$ and for $\alpha_i, \beta_j \in \mathbb{R}_+$ $(i = 1, 2, ..., p; j = 1, 2, ..., q)$; the H–function $H^{m,n}_{p,q}(z)$ is defined via a Mellin–Barnes type integral in the form

$$\mathbf{H}^{m,n}_{p,q}(z) = \mathbf{H}^{m,n}_{p,q}\left[z \,\middle|\, \begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}^{m,n}_{p,q}(s) z^{-s} ds, \qquad (8)$$

where

$$\mathcal{H}^{m,n}_{p,q}(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)}.$$

Fractional Euler–Poisson–Darboux equation

Let

$$a^* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j,$$

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i,$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}.$$

Then the H-function $\mathbf{H}_{p,q}^{m,n}(z)$ makes sense in the following case: $\Delta > 0$, $z \neq 0$, $\mathcal{L} = \mathcal{L}_{-\infty}$ is a left loop situated in a horizontal strip starting at the point $-\infty + i\varphi_1$ and terminating at the point $-\infty + i\varphi_2$ with $-\infty < \varphi_1 < \varphi_2 < +\infty$.

Fractional Euler–Poisson–Darboux equation

In

- Kilbas AA., Srivastava HM. & Trujillo JJ. Theory and applications of fractional differential equations. Elsevier: Amsterdam; 2006

the solution to the Cauchy problem

$$({}^{C}D_{0+}^{2\alpha}u)(x,t) = \lambda^2 \frac{\partial^2 u}{\partial x^2}, \qquad x \in \mathbb{R}, \qquad t > 0, \qquad \lambda > 0, \quad (9)$$

$$u(x,0) = f(x), \qquad 0 < \alpha \leq 1/2 \quad (10)$$

was given in the form

$$u(x,t) = \int_{-\infty}^{\infty} G^{\alpha}(x-\xi,t) f(\xi) d\xi. \quad (11)$$

Fractional Euler–Poisson–Darboux equation

In (11)

$$G^{\alpha}(x,t) = \frac{1}{2\lambda} t^{-\alpha} \varphi\left(-\alpha, 1-\alpha; -\frac{|x|}{\lambda} t^{-\alpha}\right). \tag{12}$$

Let $z, \rho, \beta \in \mathbb{C}$. Function $\varphi(\rho, \beta; z)$ is defined by the series

$$\varphi(\rho, \beta; z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\rho k + \beta)} \frac{z^k}{k!}. \tag{13}$$

If $\rho > -1$, the series in (13) is absolutely convergent for all $z \in \mathbb{C}$, while for $\rho = -1$ this series is absolutely convergent for $|z| < 1$ and for $|z| = 1$ and $\mathrm{Re}\, \beta > -1$. Moreover, for $\rho > -1$, $\varphi(\rho, \beta; z)$ is an entire function of z.

Fractional Euler–Poisson–Darboux equation

When $\gamma = 0$ instead of (5)–(6) we obtain (9)–(10) and (7) for $\gamma = 0$ is

$$u(x,t) = \int_{-\infty}^{\infty} G_0^{\alpha}(x-\xi,t) f(\xi) d\xi,$$

where

$$G_0^{\alpha}(x,t) = \frac{\Gamma\left(\frac{1}{2}\right)}{\lambda\sqrt{\pi}2} t^{-\alpha} \mathbf{H}_{1,3}^{2,0}\left[\frac{|x|}{\lambda} t^{-\alpha} \middle| \begin{array}{c} \left(1-\frac{\alpha}{2},\frac{\alpha}{2}\right) \\ \left(1-\frac{\alpha}{2},\frac{\alpha}{2}\right),(0,1),(\alpha,-\alpha) \end{array}\right] =$$

$$= \frac{1}{2\lambda} t^{-\alpha} \mathbf{H}_{0,2}^{1,0}\left[\frac{|x|}{\lambda} t^{-\alpha} \middle| \begin{array}{c} - \\ (0,1),(\alpha,-\alpha) \end{array}\right]$$

which coincides with (12). Here we applied the next connection between $\varphi(\rho,\beta;z)$ and $\mathbf{H}_{p,q}^{m,n}(z)$

$$\varphi(\rho,\beta;z) = \mathbf{H}_{0,2}^{1,0}\left[-z \middle| \begin{array}{c} - \\ (0,1),(1-\beta,\rho) \end{array}\right]. \tag{14}$$

Another model of random walks. Short history

Let us start from a very short historical overview. Since the founding of the “Educational Times”, in 1847, under the heading, and then in a separate edition of “Mathematical Questions”, a large number of problems have been associated with probability and particularly with random walks. In 1865, M. W. Crofton

1. Crofton, M. W. "Question 1773," Mathematical Questions with Their Solutions from the Educational Times, Vol. 4 (July-Dec. 1865), publ. 1866, 71-72.

posed the problem of a traveler’s movements along a river, which is formalized as a random walk in a straight line. Probably, it was the first mathematical illustration of the random flight concept.

Another model of random walks. Short history

Later, in a letter to Nature in 1905 [1] Karl Pearson suggested to find the probability that a man will be at the distance between r and $r + dr$ from the origin O after n displacements if he started to move from a point O and walks one yard in a straight line then he turns through any angle whatever and walks another yard in a second straight line ets. This random walk problem has attracted the interest of many researchers. In particular Lord Rayleigh answered that he solved the problem of random walk in the context of sound waves spreading in 1880 [2].

1. Pearson K.A., The problem of the random walk, Nature, July 27, 1905, 294.
2. Rayleigh (J. W. STRUTT), Baron "On the restdtant of a large number of vibrations of the same pitch and of arbitrary phase," The London, Edinburgh, and Dublin Philosophical Magazine, Set. 5, Vol. 10 (1880), 73-78.

Another model of random walks. Short history

Based on Lord Rayleigh's conceptions Jan Cornelis Kluyver in 1905 [1] proposed the general solution of the random walk problem in the terms of certain definite integrals, involving Bessel functions. Karl Pearson with his assistant in 1906 wrote a detailed study of migration [2] using the Kluyver approach. In the article [2] we can see how difficult it was to study the proposed model containing integrals of Bessel functions without using computer systems and having a limited mathematical apparatus of special functions. The authors used graphical methods and calculated the characteristics of interest to them approximately manually using power series.

1. Kluyver J. C. "A local probability problem," Proceedings of the SectiOn of Sciences, Koninklijke Akademie van Wetenschappen te Amsterdam, Vol. 8 (1905), 341-350.
2. Pearson K.A., Bakeman John. Mathematical theory of random migration. London: Dulau and Co., 1906. 54 p., VI pl.

Another model of random walks. Short history

In all of these papers, it was a question of walks on the plane. Watson [1], p.460-462 obtained a generalization of the random walk model to the case of an arbitrary number of space dimensions.

1. Watson, G. N. A Treatise on the Theory of Bessel Functions, 2nd ed., Cambridge, 1952.

A more detailed historical overview is given in the papers

1. Dutka, J. On the problem of random flights. Arch. Hist. Exact Sci. 32, 351–375 (1985).
2. Chandrasekhar S. Stochastic problems in physics and astronomy. Rev. Mod. Phys. 15:1-89. 1943.

Applications

Random walk introduced by Pearson (Pearson walks) has a large number of applications. The British theoretical physicist Lord Rayleigh studied the problem of the theory of sound, which is mathematically equivalent to the problem of a random walk [1]. He considered a set of oscillations, each with a unit amplitude, the same frequency and an arbitrary phase, and posed the problem of finding the distribution of the resulting intensity. Nobel laureate Ronald Ross [2] presented a diffusion model of the random migration of mosquitoes when he studied the laws of the spread of malaria.

1. Rayleigh (J. W. STRUTT), Baron "On the restdtant of a large number of vibrations of the same pitch and of arbitrary phase," The London, Edinburgh, and Dublin Philosophical Magazine, Set. 5, Vol. 10 (1880), 73-78.
2. Ross Ronald. On the logical basis of the sanitary policy of mosquito reduction // Proceedings of the Congress of Arts and Sciences, USA, St Louis. 1904. V. 6. P. 89.

Applications

Pearson's paper [1] introduced the application of random walks to the description of migration models. Another interesting application where a special case of the random walk model appeared is a description of polymer conflguratlons [2], p. 142, formula 4.28b.

1. Pearson K.A., Bakeman John. Mathematical theory of random migration. London: Dulau and Co., 1906. 54 p., VI pl.
2. M. V. Volkenstein The Configurational Statistics of Polymeric Chains. USSR Academy of Sciences. 1959. 466 p.

Applications

As the next application of Pearson random walks is in the analysis of narrowband signals in noise [1]. Next remarkable application is motion of microorganisms on surfaces [2,3]. Also such model appeared in crystallography [4].

1. R. Barakat, J. E. Cole III, Statistical properties of n random sinusoidal waves in additive Gaussian noise. Journal of Sound and Vibration (1979) 62(3), 365-377
2. H. C. Berg, D. A. Brown Chemotaxis in Escherichia coli analysed by Three-dimensional Tracking, Nature. 239, pages 500–504 (1972).
3. R. M. Macnab, D. E. Koshland Jr. The gradient-sensing mechanism in bacterial chemotaxis. Proc Natl Acad Sci USA. 1972 Sep;69(9):2509-12.
4. R. Srinivisan, S. Parthasarathy, Some Statistical Applications in X-ray Crystallography (Pergamon Press, London, 1976).

Novelity

Historical and modern works on this topic mainly concern the discrete case of a random walk, in particular, for example, when a walk occurs along a lattice oriented parallel to the rectangular coordinate axes of a k-dimensional Euclidean space. Relatively little attention has been paid to the continuous case of a random walk, in which the direction of a wandering object can change continuously from one step to the next. Why is this happening? The fundamental difference between the mathematical model of a random walk with an arbitrary, continuously changing angle of direction of a moving object from a walk on a grid is to use a generalized translation instead of the usual one.

Novelity

The generalized translation is a singular integral operator, and the corresponding differential equations contain the Bessel operator instead of the usual derivative. Therefore, the reason that the model proposed by Rayleigh and Pearson did not receive rich theoretical development was the lack of a suitable mathematical apparatus aimed at working with a generalized translation. Here, we present a mathematical description of a continuous random walk model, in which the direction of a wandering object can continuously change from one step to another, and show that it is supported by experimental data.

Quantitative model of random walks

Let agents $A = \{a_1, ..., a_s\}$ are concentrated at the origin. So at the time t_0 agent a_i, $i = \overline{1, s}$ have position X_0. Then it starts to jumps from the center and at times t_1, t_2,...,t_n it has displacements X_1, X_2,...,X_n. The resultant at t_n is $S_n = X_0 + \sum\limits_{m=1}^{n} X_m$. The displacements are assumed to be independent and the probability density of X_m is $p_m(X_m)$ and the probability density of S_n is required to be found. We start from finding of the probability $Pr(|S_n| \leq r)$ that S_n lies inside or on a circle of radius r.

Quantitative model of random walks

Theorem

Let l_m denotes the length of the m–th jump, $m = 1, ..., n$. Let $Pr(|S_n| \leq r) = Pr(|S_n| \leq r; l_1, l_2, ..., l_n)$ is the probability that S_n lies inside or on a circle of radius r centred at origin O. Then the next formula for $Pr(|S_n| \leq r)$ is valid

$$Pr(|S_n| \leq r) = \chi(r)\frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int\limits_0^\infty J_{\frac{\nu}{2}}(rt) \prod_{m=1}^{n} j_{\frac{\nu}{2}-1}(l_m t) t^{\frac{\nu}{2}-1} dt, \quad (15)$$

where J_η is a Bessel function of the first kind,
$j_\eta(x) = \frac{2^\eta \Gamma(\eta+1)}{x^\eta} J_\eta(x)$ is a normalize Bessel function,

$$\chi(r) = \begin{cases} 1 & \text{if } r \neq |S_n|; \\ \frac{1}{2} & \text{if } r = |S_n|. \end{cases}$$

Quantitative model of random walks

Proof. In this problem we suppose that all values of θ_m, $m=\overline{1,n}$ are not equally likely and depend on parameter $\nu>1$. It expresses in that the element of angle θ_m multiplied by the factor $\sin^{\nu-2}\theta_m$ and θ_m varies from 0 to π. Taking into account the formula

$$\int\limits_0^\pi \sin^{\nu-2} t dt = \frac{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}$$

we get that the probability that an agent will be in a circle of radius r after the n-th jump for $n\geq 2$ is the $(n-1)$-tuple integral of the form

$$Pr(|S_n|\leq r)=\left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)}\right)^{n-1}\times$$

$$\times\int\limits_0^\pi \sin^{\nu-2}\theta_1 d\theta_1 \int\limits_0^\pi \sin^{\nu-2}\theta_2 d\theta_2 ... \int\limits_0^\pi \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \int \sin^{\nu-2}\theta_{n-1}\, d\theta_{n-1}. \tag{16}$$

Quantitative model of random walks

Proof.

In (16) the integration with respect to θ_{n-1} extends over the values of θ_{n-1} which make $|S_n| \leq r$. Formula (16) generalizes the formula from [1], p.461 where is $\nu = p \geq 2$ only natural numbers.
When using the Weber–Schafheitlin formula from [1] for $\nu > 0$ of the form

$$\int\limits_0^\infty J_{\nu-1}(st) J_\nu(rt)\, dt = \begin{cases} \frac{s^{\nu-1}}{r^\nu} & \text{for } 0 < s < r; \\ \frac{1}{2r} & \text{for } s = r; \\ 0 & \text{for } s > r. \end{cases} \tag{17}$$

1. Watson, G. N. A Treatise on the Theory of Bessel Functions, 2nd ed., Cambridge, 1952.

Quantitative model of random walks

Proof. So, if discontinuous factor

$$\chi(r)\frac{r^{\frac{\nu}{2}}}{|S_n|^{\frac{\nu}{2}-1}}\int\limits_0^\infty J_{\frac{\nu}{2}-1}(|S_n|t)J_{\frac{\nu}{2}}(rt)dt=\begin{cases}1 & \text{for } |S_n|\le r;\\ 0 & \text{for } |S_n|>r,\end{cases} \tag{18}$$

where

$$\chi(r)=\begin{cases}1 & \text{if } r\neq|S_n|;\\ \frac{1}{2} & \text{if } r=|S_n|\end{cases}$$

is inserted in the $(n-1)$-tuple integral $Pr(|S_n|\le r)$, the range of values of θ_m may be taken to be from 0 to π.

Quantitative model of random walks

Proof. We obtain

$$Pr(|S_n| \leq r) = \chi(r) r^{\frac{\nu}{2}} \left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)} \right)^{n-1} \times$$

$$\times \int_0^\pi \sin^{\nu-2}\theta_1 d\theta_1 \int_0^\pi \sin^{\nu-2}\theta_2 d\theta_2 \dots \int_0^\pi \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \times$$

$$\times \int_0^\pi \sin^{\nu-2}\theta_{n-1} |S_n|^{1-\frac{\nu}{2}} \, d\theta_{n-1} \int_0^\infty J_{\frac{\nu}{2}-1}(|S_n|t) J_{\frac{\nu}{2}}(rt) dt =$$

$$= \chi(r) r^{\frac{\nu}{2}} \left(\frac{\Gamma\left(\frac{\nu}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)} \right)^{n-1} \int_0^\pi \sin^{\nu-2}\theta_1 d\theta_1 \int_0^\pi \sin^{\nu-2}\theta_2 d\theta_2 \times \dots$$

$$\dots \times \int_0^\pi \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \int_0^\infty J_{\frac{\nu}{2}}(rt) dt \int_0^\pi \frac{J_{\frac{\nu}{2}-1}(|S_n|t)}{|S_n|^{\frac{\nu}{2}-1}} \sin^{\nu-2}\theta_{n-1} \, d\theta_{n-1}.$$

Quantitative model of random walks

Proof. Using the definition of normalized Bessel function of the first kind j_ν

$$j_\eta(x) = \frac{2^\eta \Gamma(\eta+1)}{x^\eta} \, J_\eta(x), \tag{19}$$

we can write

$$Pr(|S_n| \leq r) = \chi(r) 2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}} \frac{\Gamma^{n-2}\left(\frac{\nu}{2}\right)}{\left(\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)\right)^{n-1}} \times$$

$$\times \int\limits_0^\pi \sin^{\nu-2}\theta_1 d\theta_1 \int\limits_0^\pi \sin^{\nu-2}\theta_2 d\theta_2 ... \int\limits_0^\pi \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \times$$

$$\times \int\limits_0^\infty J_{\frac{\nu}{2}}(rt) t^{\frac{\nu}{2}-1} dt \int\limits_0^\pi j_{\frac{\nu}{2}-1}(|S_n| t) \, \sin^{\nu-2}\theta_{n-1} \, d\theta_{n-1}.$$

Quantitative model of random walks

Proof.

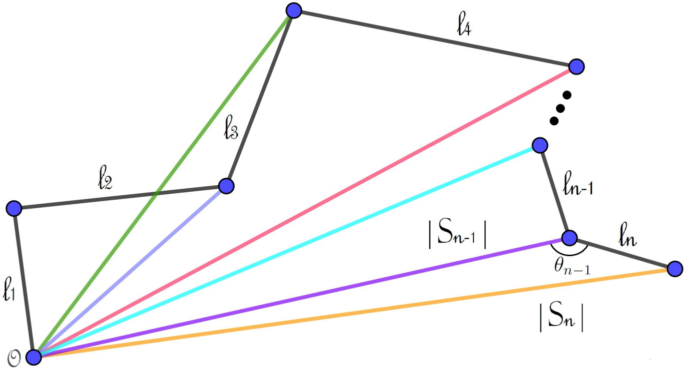

Figure 1: Migration scheme on the n-th jump.

Quantitative model of random walks

Proof. Since (see Figure 1)

$$|S_n| = \sqrt{|S_{n-1}|^2 - 2|S_{n-1}|l_n \cos\theta_{n-1} + l_n^2}$$

and for integral by θ_{n-1} we obtain

$$\int_0^{\pi} j_{\frac{\nu}{2}-1}(|S_n|t)\,\sin^{\nu-2}\theta_{n-1}\,d\theta_{n-1} =$$

$$\int_0^{\pi} j_{\frac{\nu}{2}-1}\left(\sqrt{|S_{n-1}|^2 - 2|S_{n-1}|l_n\cos\theta_{n-1} + l_n^2}\cdot t\right)\sin^{\nu-2}\theta_{n-1}\,d\theta_{n-1} =$$

$$= \frac{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}\,{}^{\nu-1}T_{l_n}^{|S_{n-1}|} j_{\frac{\nu}{2}-1}(l_n t)$$

where $({}^{\gamma}T_x^y f)(x)$ is generalized translation of the form

$$({}^{\gamma}T_x^y f)(x) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)}\int_0^{\pi} f\left(\sqrt{x^2+y^2-2xy\cos\theta}\right)\sin^{\gamma-1}\theta\,d\theta, \qquad \gamma > 0.$$

Quantitative model of random walks

Proof. The formula

$$ {}^{\gamma}T_x^y j_{\frac{\gamma-1}{2}}(x\xi) = j_{\frac{\gamma-1}{2}}(x\xi)\, j_{\frac{\gamma-1}{2}}(y\xi) $$

is known (see [1]). Therefore,

$$ \int_0^{\pi} j_{\frac{\nu}{2}-1}\left(\sqrt{|S_{n-1}|^2 - 2|S_{n-1}|l_n \cos\theta_{n-1} + l_n^2}\cdot t\right) \sin^{\nu-2}\theta_{n-1}\, d\theta_{n-1} = $$

$$ = \frac{\sqrt{\pi}\,\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} j_{\frac{\nu}{2}-1}(l_n t) j_{\frac{\nu}{2}-1}(|S_{n-1}|t). $$

1. Levitan, B.M., 1951. Expansion in Fourier Series and Integrals with Bessel Functions. Uspekhi Mat. Nauk, 6, 2 (42), 102–143.

Quantitative model of random walks

Proof. So we obtain

$$Pr(|S_n| \leq r) =$$

$$= \chi(r) 2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}} \frac{\Gamma^{n-2}\left(\frac{\nu}{2}\right)}{\left(\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)\right)^{n-1}} \frac{\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \int\limits_0^{\pi} \sin^{\nu-2}\theta_1 d\theta_1 \int\limits_0^{\pi} \sin^{\nu-2}\theta_2 d\theta_2 \times \ldots$$

$$\ldots \times \int\limits_0^{\pi} \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \int\limits_0^{\infty} J_{\frac{\nu}{2}}(rt) j_{\frac{\nu}{2}-1}(l_n t) j_{\frac{\nu}{2}-1}(|S_{n-1}|t) t^{\frac{\nu}{2}-1} dt.$$

Quantitative model of random walks

And

$$Pr(|S_n| \leq r) =$$

$$= \chi(r) 2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}} \frac{\Gamma^{n-3}\left(\frac{\nu}{2}\right)}{\left(\sqrt{\pi}\Gamma\left(\frac{\nu-1}{2}\right)\right)^{n-2}} \int\limits_0^{\pi} \sin^{\nu-2}\theta_1 d\theta_1 \int\limits_0^{\pi} \sin^{\nu-2}\theta_2 d\theta_2 \times \ldots$$

$$\ldots \times \int\limits_0^{\pi} \sin^{\nu-2}\theta_{n-2} d\theta_{n-2} \int\limits_0^{\infty} J_{\frac{\nu}{2}}(rt) j_{\frac{\nu}{2}-1}(l_n t) j_{\frac{\nu}{2}-1}(|S_{n-1}|t) t^{\frac{\nu}{2}-1} dt.$$

By repetitions of this process we get (15):

$$Pr(|S_n| \leq r) = \chi(r) \frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int\limits_0^{\infty} J_{\frac{\nu}{2}}(rt) \prod_{m=1}^{n} j_{\frac{\nu}{2}-1}(l_m t) t^{\frac{\nu}{2}-1} dt.$$

What was required to prove.

Quantitative model of random walks

Since

$$\frac{d}{dr} r^{\frac{\nu}{2}} J_{\frac{\nu}{2}}(rt) = t r^{\frac{\nu}{2}} J_{\frac{\nu}{2}-1}(rt)$$

on differentiating $Pr(|S_n| \leq r)$ with respect to r, one gets the probability density

$$f_n(r) = \chi(r) \frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^\infty J_{\frac{\nu}{2}-1}(rt) \prod_{m=1}^{n} j_{\frac{\nu}{2}-1}(l_m t) t^{\frac{\nu}{2}} dt. \tag{20}$$

Quantitative model of random walks

Since We can calculate the integral in (20) using formula 2.12.44.5 from [1], p. 207. We get

$$f_n(r) = \chi(r) \frac{2r^{\frac{\nu}{2}-2}}{\Gamma\left(\frac{\nu}{2}\right)} F_C^{(n)} \left(\frac{\nu}{2}, 1; \underbrace{\frac{\nu}{2}, ..., \frac{\nu}{2}}_{n}; \frac{l_1^2}{r^2}, ..., \frac{l_n^2}{r^2} \right).$$

$$F_C^{(n)}(a, b; c_1, ..., c_n; z_1, ..., z_n) = \sum_{k_1,...,k_n=0}^{\infty} \frac{(a)_{k_1+...+k_n}(b)_{k_1+...+k_n}}{(c_1)_{k_1}...(c_n)_{k_n}} \frac{z_1^{k_1}...z_n^{k_n}}{k_1!...k_n!},$$

is the Lauricella function, $(z)_n = z(z+1)...(z+n-1)$, $n = 1, 2, ...$, $(z)_0 \equiv 1$ is the Pochhammer symbol.

1. Prudnikov, A.P., Brychkov, Yu.A., Marichev, O.I., 1990. Integrals and Series, Vol. 2, Special Functions. Gordon & Breach Sci. Publ., New York.

Quantitative model of random walks

If $l_1 = l_2 = ... = l_n = l$ then

$$Pr(|S_n| \leq r) = \chi(r) \frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} J_{\frac{\nu}{2}}(rt) j^n_{\frac{\nu}{2}-1}(lt) t^{\frac{\nu}{2}-1} dt. \qquad (21)$$

Putting $l = vl_1$ we get

$$Pr(|S_n| \leq r) = \chi(r) \frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} J_{\frac{\nu}{2}}(rt) j^n_{\frac{\nu}{2}-1}(vl_1 t) t^{\frac{\nu}{2}-1} dt = \{vt = \tau\} =$$

$$= \frac{1}{v} \chi(r) \frac{2^{1-\frac{\nu}{2}} r^{\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \int_0^{\infty} J_{\frac{\nu}{2}}\left(r\frac{\tau}{v}\right) j^n_{\frac{\nu}{2}-1}(l_1\tau) t^{\frac{\nu}{2}-1} d\tau. \qquad (22)$$

Special cases of quantitative model

Here we compare the resulting in previous subsection model with previously known models.

1. If $\nu = 2$ and $l_1 = l_2 = \ldots = l_n = l$ then we get Person–Raylleigh model

$$Pr(|S_n| \leq r) = \chi(r) r \int_0^\infty J_1(rt) J_0^n(lt)\, dt,$$

$$f_n(r) = \chi(r) r \int_0^\infty t J_0(rt) J_0^n(lt)\, dt.$$

Special cases of quantitative model

1. For $\nu = 2$ we obtain Kluyver model for the case when agents moving by plain and the choice of the angle θ_m between $-\pi$ and π are equally probable. In this model the probability that the distance from the starting point will be less or equal than r after n flights is,

$$Pr(|S_n| \leq r) = \chi(r) r \int_0^\infty J_1(rt) \prod_{m=1}^{n} J_0(l_m t) dt.$$

and the probability density is

$$f_n(r) = \chi(r) r \int_0^\infty t J_0(rt) \prod_{m=1}^{n} J_0(l_m t) dt.$$

Here we notice that $j_0(x) = J_0(x)$.

Special cases of quantitative model

1. For $\nu = k \in \mathbb{N}$, $k \geq 3$ in (15) we obtain Watson model This case corresponds to the problem for space of k dimensions. If generalised polar coordinates (in which θ_m is regarded as a co-latitude) are used, the element of generalised solid angle contains θ_m only by the factor $\sin^{p-2}\theta_m d\theta_m$ and θ_m varies from 0 to π. The symmetry with respect to the polar axis enables us to disregard the factor depending on the longitudes.

THANK YOU FOR ATTENTION.