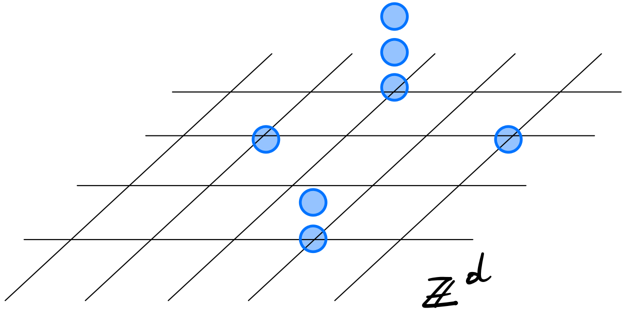
Hydrodynamics for the Partial Exclusion Process in Random Environment

FD2W01, Isaac Newton Institute

25th of February 2022

Simone Floreani
(TU Delft)

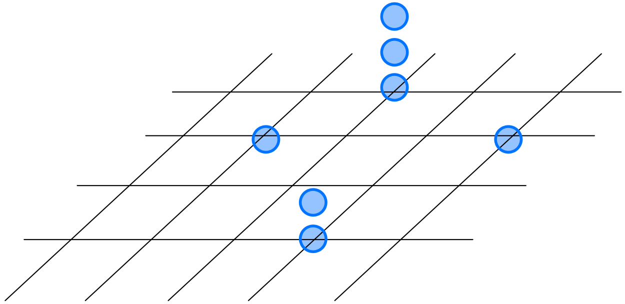
Interacting Particle Systems



$\mathbb{Z}^d$

● = PARTICLE (RANDOM WALK)

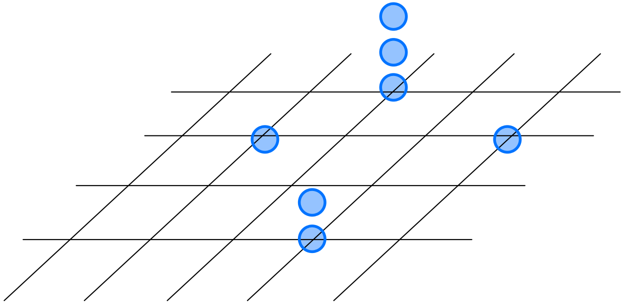
$$\eta_t = \left(\eta_t(x)\right)_{x \in \mathbb{Z}^d}$$ (MARKOV PROCESS)



= PARTICLE (RANDOM WALK)

$$\eta_t = (\eta_t(x))_{x \in \mathbb{Z}^d}$$ (MARKOV PROCESS)

FROM MICRO TO MACRO

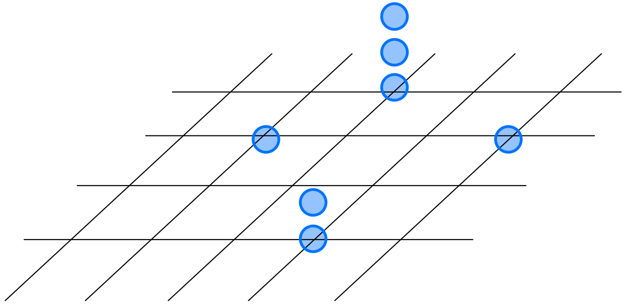


= PARTICLE (RANDOM WALK)

$$\eta_t = \left(\eta_t(x)\right)_{x \in \mathbb{Z}^d} \quad \text{(MARKOV PROCESS)}$$

FROM MICRO TO MACRO

$$\mathbb{X}^N_t := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_{tN^2}(x)\, \delta_{\frac{x}{N}}$$



$\bullet$ = PARTICLE (RANDOM WALK)

$\eta_t = (\eta_t(x))_{x \in \mathbb{Z}^d}$ (MARKOV PROCESS)

FROM MICRO TO MACRO

$$\mathbb{X}^N_t := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_{tN^2}(x)\, \delta_{\frac{x}{N}} \xrightarrow[\text{HYDRODYNAMIC LIMIT}]{N \to \infty} \underbrace{\rho_t(u)\, du}_{\text{SOLUTION OF SOME PDEs}}$$

My Research

F. Redig
(TU Delft)

F. den Hollander
(Leiden Uni.)

C. Giardina
(Modena Uni.)

me
(TU Delft)

S. Nandan
(Leiden Uni.)

Stochastic Duality

STOCHASTIC DUALITY

- MARKOV PROCESS $\longrightarrow$ DUAL MARKOV PROCESS

η_t $\qquad\qquad\qquad$ ξ_t

STOCHASTIC DUALITY

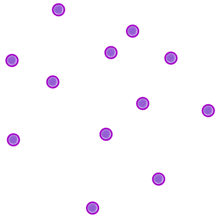
- MARKOV PROCESS $\longrightarrow$ DUAL MARKOV PROCESS

η_t $\qquad\qquad$ ξ_t

$$\boxed{\mathbb{E}_\eta\left[D(\eta_t,\xi)\right] = \hat{\mathbb{E}}_\xi\left[D(\eta,\xi_t)\right]}$$

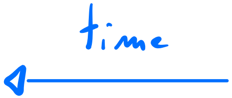
$$\boxed{\forall\ t \geq 0,\ \eta,\ \xi}$$

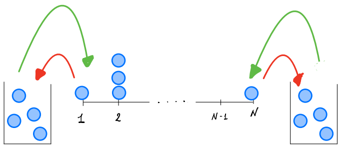
SIMPLIFICATIONS DUE TO DUALITY

FROM  TO 

FROM $\mathbb{R}^d$ TO $\mathbb{Z}^d$

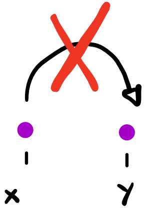
FROM 

TO 

FROM 

TO 

EXAMPLE: SIMPLE SYMMETRIC EXCLUSION PROCESS (SEP)



at Rate $(1-\eta(y))$

EXAMPLE: SIMPLE SYMMETRIC EXCLUSION PROCESS (SEP)

at Rate $(1-\eta(y))$

$$\mathcal{X} := \{0,1\}^{\mathbb{Z}^d}$$

$$\mathcal{L} f(\eta) = \sum_{\substack{x,y\in\mathbb{Z}^d \\ |x-y|=1}} \Big\{ \eta(x)\,(1-\eta(y))\,\big(f(\eta^{x,y}) - f(\eta)\big) + \eta(y)\,(1-\eta(x))\,\big(f(\eta^{y,x}) - f(\eta)\big) \Big\}$$

Example: Simple Symmetric Exclusion Process (SEP)

at Rate $(1 - \eta(y))$

Duality for SEP

$$\mathbb{E}_{\eta_0}[\eta_t(x)] = \mathbb{E}_x^{RW}[\eta_0(X_t)]$$

$$\mathbb{E}_{\eta_0}[\eta_t(x)] = \mathbb{E}_x^{RW}[\eta_0(X_t)]$$

HYDRODYNAMIC LIMIT

$$\mathbb{X}_t^N := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_{tN^2}(x)\, \delta_{\frac{x}{N}}$$

$$\mathbb{E}_{\eta_0}[\eta_t(x)] = \mathbb{E}_x^{RW}[\eta_0(X_t)]$$

HYDRODYNAMIC LIMIT

$$\mathbb{X}_t^N := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_{tN^2}(x)\, \delta_{\frac{x}{N}}$$

- $$\mathbb{E}_\eta[\mathbb{X}_t^N] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\eta[\eta_{tN^2}(x)]\, \delta_{\frac{x}{N}}$$

$$\mathbb{E}_{\eta_0}[\eta_t(x)] = \mathbb{E}_x^{RW}[\eta_0(X_t)]$$

HYDRODYNAMIC LIMIT

$$\mathbb{X}_t^N := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_{tN^2}(x) \, \delta_{\frac{x}{N}}$$

- $$\mathbb{E}_\eta[\mathbb{X}_t^N] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\eta[\eta_{tN^2}(x)] \, \delta_{\frac{x}{N}}$$

(DUALITY) $$= \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_x^{RW}[\eta_0(X_{tN^2})] \, \delta_{\frac{x}{N}}$$

$$\mathbb{E}_{\eta_0}[\eta_t(x)] = \mathbb{E}_x^{RW}[\eta_0(X_t)]$$

Hydrodynamic Limit

$$\mathbb{X}_t^N := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_{tN^2}(x) \, \delta_{\frac{x}{N}}$$

- $$\mathbb{E}_\eta[\mathbb{X}_t^N] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\eta[\eta_{tN^2}(x)] \, \delta_{\frac{x}{N}}$$

(DUALITY) $$= \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_x^{RW}[\eta_0(X_{tN^2})] \, \delta_{\frac{x}{N}}$$

$\left(\eta_0 \overset{N}{\rightarrow} \bar{\rho}\right)$ + I.P. $$\approx \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\frac{x}{N}}^{B.M.}[\bar{\rho}(B_t)] \, \delta_{\frac{x}{N}}$$

HYDRODYNAMIC LIMIT

$$\mathbb{X}_t^N(G) := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_{tN^2}(x)\, G\left(\frac{x}{N}\right)$$

$$G \in C_c(\mathbb{R}^d)$$

- $$\mathbb{E}_\eta\left[\mathbb{X}_t^N(G)\right] \approx \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}^{B.M.}_{\frac{x}{N}}\left[\bar\rho(B_t)\right] G\left(\frac{x}{N}\right)$$

$$\downarrow$$

$$\int_{\mathbb{R}^d} \rho_t(u)\, G(u)\, du$$

$$\boxed{\begin{cases} \partial_t \rho_t = \frac{1}{2} \Delta \rho_t \\ \rho_0 = \bar\rho \end{cases}}$$

THEOREM

ASSUME $\quad X_0^N \xRightarrow[N]{LAW} \bar{\rho}(u)\,du$

THEN $\forall\, T > 0$

$$\underbrace{\left\{ X_t^N,\ t \in [0,T] \right\}}_{D([0,T],\, \mathcal{S}'(\mathbb{R}^d))} \xRightarrow[N]{LAW} \underbrace{\left\{ \rho_t(u)\,du\ ,\ t \in [0,T] \right\}}_{\begin{cases} \partial_t \rho_t = \frac{1}{2} \Delta \rho_t \\ \rho_0 = \bar{\rho} \end{cases}}$$

STOCHASTIC DUALITY

&

STOCHASTIC HOMOGENIZATION

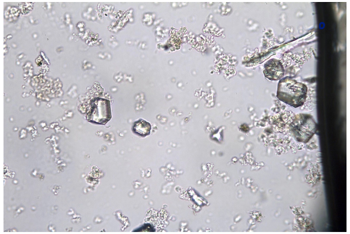
FOR IPS IN RANDOM ENVIRONMENT

STOCHASTIC DUALITY

&

STOCHASTIC HOMOGENIZATION

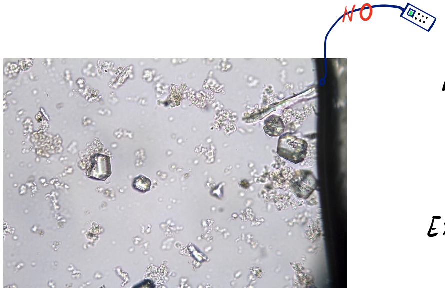
FOR IPS IN RANDOM ENVIRONMENT



IMPURITIES

STOCHASTIC DUALITY
&
STOCHASTIC HOMOGENIZATION

FOR IPS IN RANDOM ENVIRONMENT



NO MEASURES

$\Downarrow$

EXTRA RANDOMNESS

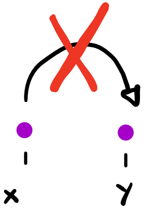
IMPURITIES

STOCHASTIC DUALITY

&

STOCHASTIC HOMOGENIZATION

FOR IPS IN RANDOM ENVIRONMENT



at Rate $\omega_{xy}(1-\eta(y))$

New Random Environment for the Exclusion Process

with

F. Sau (IST Austria)

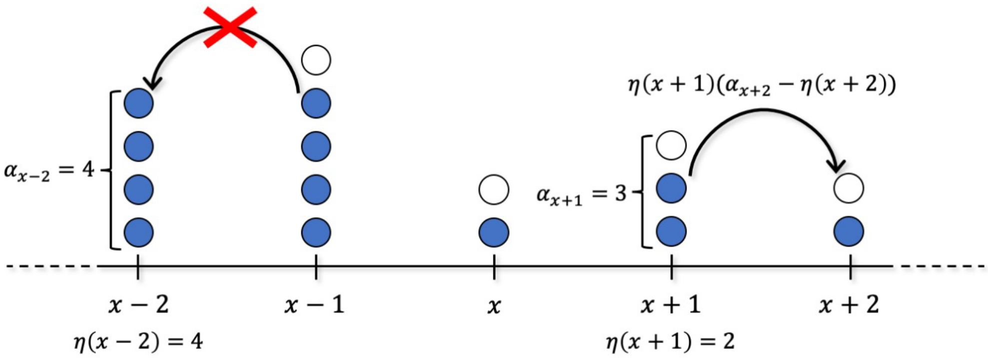
&

F. Redig (TU Delft)

PARTIAL EXCLUSION PROCESS in RANDOM ENVIRONMENT

WITH F. REDIG (TU DELFT) & F. SAU (IST AUSTRIA)

α_x = RANDOM MAXIMAL OCCUPANCY



SEP (α)

PARTIAL EXCLUSION PROCESS in RANDOM ENVIRONMENT

WITH F. REDIG (TU DELFT) & F. SAU (IST AUSTRIA)

$$\mathcal{X}^{\alpha} = \prod_{x \in \mathbb{Z}^d} \{0, \dots, \alpha_x\}$$

$$\mathcal{L}^{\alpha} f(\eta) = \sum_{\substack{x, y \in \mathbb{Z}^d \\ |x-y|=1}} \left\{ \begin{array}{l} \eta(x)\,(\alpha_y - \eta(y))\,\left(f(\eta^{x,y}) - f(\eta)\right) \\ + \\ \eta(y)\,(\alpha_x - \eta(x))\,\left(f(\eta^{y,x}) - f(\eta)\right) \end{array} \right\}$$

DUALITY

SEP(α)

$$D(x,\eta) = \frac{\eta(x)}{\alpha_x}$$

RW(α)

$(\eta_t)_{t\geq 0}$ $\longleftrightarrow$ $(X_t^{\alpha,x})_{t\geq 0}$

$$\boxed{A^{\alpha} g(x) = \sum_{y:|y-x|=1} \alpha_y \left(g(y) - g(x) \right)}$$

$(\mathcal{J}_t^{\alpha,RW})_{t\geq 0}$ SEMIGROUP

$\alpha = (\alpha_x)_{x\in\mathbb{Z}^d}$ REV. MEAS.

HDL for SEP(α)

(F., REDIG, SAU)

ASSUMPTIONS: $\boldsymbol{\alpha} = (\alpha_x)_{x \in \mathbb{Z}^d} \sim \mathbb{Q}$

s.t. i) $\forall_x \quad 1 \leq \alpha_x \leq C \quad \mathbb{Q}$ a.s.

ii) $\mathbb{Q}$ is ERGODIC UNDER TRANSLATIONS IN $\mathbb{Z}^d$

HDL for SEP(α) (F., REDIG, SAU)

ASSUMPTIONS: $\boldsymbol{\alpha} = (\alpha_x)_{x \in \mathbb{Z}^d} \sim \mathbb{Q}$

s.t. i) $\forall x \quad 1 \leq \alpha_x \leq C \quad \mathbb{Q}$ a.s.

ii) $\mathbb{Q}$ IS ERGODIC UNDER TRANSLATIONS IN $\mathbb{Z}^d$

THEN $\exists \ \Sigma \in \mathbb{R}^{d \times d}$ SYM. & POS. DEF.

S.T. $\mathbb{Q}$ - a.e. $\boldsymbol{\alpha} = (\alpha_x)_{x \in \mathbb{Z}^d}$

$$\left(\mathbb{X}_t^{N,\boldsymbol{\alpha}} := \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_{tN^2}(x) \, \delta_{\frac{x}{N}} \right)_{t \in [0,T]} \Longrightarrow \left(\rho_t(u) \, du \right)_{t \in [0,T]}$$

$$\partial_t \rho_t = \nabla \cdot (\Sigma \nabla) \rho_t$$

STOCHASTIC HOMOGENIZATION FOR SEP(α)

$\xrightarrow{\text{DUALITY}}$

STOCHASTIC HOMOGENIZATION FOR RW(α)

STOCHASTIC HOMOGENIZATION

$G \in C_c(\mathbb{R}^d)$

$\alpha = (\alpha_x)_{x \in \mathbb{Z}^d} \to \Sigma$

$$\mathbb{E}_{\eta_0}\left[\mathbb{X}_t^{N,\alpha}(G) \right] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\eta_0}\left[\eta_{tN^2}(x) \right] G\left(\frac{x}{N}\right)$$

STOCHASTIC HOMOGENIZATION

$G \in C_c(\mathbb{R}^d)$ $\qquad\qquad$ $\alpha = (\alpha_x)_{x \in \mathbb{Z}^d} \to \Sigma$

$$\mathbb{E}_{\eta_0}\left[\mathbb{X}_t^{N,\alpha}(G) \right] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\eta_0}\left[\frac{\eta_{tN^2}(x)}{\alpha_x} \right] G\left(\frac{x}{N}\right) \alpha_x$$

STOCHASTIC HOMOGENIZATION

$G \in C_c(\mathbb{R}^d)$ $\qquad \alpha = (\alpha_x)_{x \in \mathbb{Z}^d} \to \Sigma$

$$\mathbb{E}_{\eta_0}\left[\mathbb{X}_t^{N,\alpha}(G) \right] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\eta_0}\left[\frac{\eta_{tN^2}(x)}{\alpha_x} \right] G\left(\frac{x}{N}\right) \alpha_x$$

DUALITY $\displaystyle = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} S_{tN^2}^{\alpha, RW}\left(\frac{\eta_0(\cdot)}{\alpha_\cdot}\right)(x)\, G\left(\frac{x}{N}\right) \alpha_x$

STOCHASTIC HOMOGENIZATION

$G \in C_c(\mathbb{R}^d)$ $\qquad \alpha = (\alpha_x)_{x \in \mathbb{Z}^d} \to \Sigma$

$$\mathbb{E}_{\eta_0}\left[\mathcal{X}_t^{N,\alpha}(G) \right] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\eta_0}\left[\frac{\eta_{tN^2}(x)}{\alpha_x} \right] G\left(\frac{x}{N}\right) \alpha_x$$

DUALITY

$$= \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathcal{S}^{\alpha, RW}_{tN^2}\left(\frac{\eta_0(\cdot)}{\alpha_\cdot} \right)(x) \; G\left(\frac{x}{N}\right) \alpha_x$$

SELF-AD. IN $\ell^2(\alpha)$

$$= \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \frac{\eta_0(x)}{\alpha_x} \left(\mathcal{S}^{\alpha, RW}_{tN^2} G\left(\frac{\cdot}{N}\right) \right)(x) \; \alpha_x$$

STOCHASTIC HOMOGENIZATION

$G \in C_c(\mathbb{R}^d)$ $\qquad\qquad$ $\alpha = (\alpha_x)_{x \in \mathbb{Z}^d} \to \Sigma$

$$\mathbb{E}_{\eta_0}\left[\mathbb{X}_t^{N,\alpha}(G) \right] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathbb{E}_{\eta_0}\left[\frac{\eta_{tN^2}(x)}{\alpha_x} \right] G\left(\frac{x}{N}\right) \alpha_x$$

DUALITY

$$= \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \mathcal{S}^{\alpha, RW}_{tN^2}\left(\frac{\eta_0(\cdot)}{\alpha_\cdot} \right)(x) \; G\left(\frac{x}{N}\right) \alpha_x$$

SELF-AD. IN $\ell^2(\alpha)$

$$= \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \frac{\eta_0(x)}{\cancel{\alpha_x}} \left(\mathcal{S}^{\alpha, RW}_{tN^2} G\left(\frac{\cdot}{N}\right)\right)(x) \; \cancel{\alpha_x}$$

STOCHASTIC HOMOGENIZATION

$G \in C_c(\mathbb{R}^d)$ $\qquad\qquad$ $\alpha = (\alpha_x)_{x \in \mathbb{Z}^d} \to \Sigma$

$$\mathbb{E}_{\eta_0}\left[\mathbb{X}_t^{N,\alpha}(G) \right] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_0(x) \left(\mathcal{S}_{tN^2}^{\alpha, RW} G\left(\frac{\cdot}{N}\right) \right)(x)$$

TO PROVE :

FOR $\mathbb{Q}$-A.E. α

$$\boxed{\frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \left| \mathcal{S}_{tN^2}^{\alpha, RW} G\left(\frac{x}{N}\right) - \mathcal{S}_t^{BM^{\Sigma}} G\left(\frac{x}{N}\right) \right| \eta_0(x) \xrightarrow{N \to \infty} 0}$$

STOCHASTIC HOMOGENIZATION

$G \in C_c(\mathbb{R}^d)$ $\qquad$ $\alpha = (\alpha_x)_{x \in \mathbb{Z}^d} \to \Sigma$

$$\mathbb{E}_{\eta_0}\left[\mathbb{X}_t^{N,\alpha}(G) \right] = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_0(x) \left(\mathcal{S}_{tN^2}^{\alpha, RW} G\left(\frac{\cdot}{N}\right) \right)(x)$$

TO PROVE :

FOR $\mathbb{Q}$-A.E. α

$$\boxed{\frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \left| \mathcal{S}_{tN^2}^{\alpha, RW} G\left(\frac{x}{N}\right) - \mathcal{S}_t^{BM^{\Sigma}} G\left(\frac{x}{N}\right) \right| \cancel{\eta_0(x)} \xrightarrow[N \to \infty]{} 0}$$

$\mathbb{Q}$-A.E. α, $G \in C_c(\mathbb{R}^d)$

$$\sup_{t\in[0,T]} \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left| \mathcal{S}^{\alpha,RW}_{tN^2} G\left(\tfrac{x}{N}\right) - \mathcal{S}^{BM^{\Sigma}}_{t} G\left(\tfrac{x}{N}\right) \right| \longrightarrow 0$$

L^1 CONV.

$$\Updownarrow$$

$\forall\, u \in \mathbb{R}^d$, $\forall \{x_N\} \subset \mathbb{Z}^d$ S.T. $\frac{x_N}{N} \to u$

A.S.P. Q.I.P.

$$\left(\frac{1}{N} X^{\alpha, x_N}_{tN^2}\right)_{t\in[0,T]} \Rightarrow \left(B^{\Sigma}_t + u\right)_{t\geq 0}$$

$\mathbb{Q}$-A.E. α, $G \in C_c(\mathbb{R}^d)$

$$\sup_{t\in[0,T]} \frac{1}{N^d} \sum_{x\in\mathbb{Z}^d} \left| \mathcal{S}^{\alpha,RW}_{tN^2} G\left(\tfrac{x}{N}\right) - \mathcal{S}^{BM^\Sigma}_t G\left(\tfrac{x}{N}\right)\right| \longrightarrow 0$$

L^1 CONV.

$$\Updownarrow$$

$\forall\, u \in \mathbb{R}^d$, $\forall \{x_N\} \subset \mathbb{Z}^d$ s.t. $\frac{x_N}{N} \to u$

$$\left(\frac{1}{N} X^{\alpha, x_N}_{tN^2}\right)_{t\in[0,T]} \Longrightarrow \left(B^\Sigma_t + u\right)_{t\geq 0}$$

A.S.P.Q.I.P.

- I.P. FROM 0
- EQUIC. OF $\mathcal{S}^{\alpha,RW}_t$
- HEAT KERNEL UPPER BOUNDS

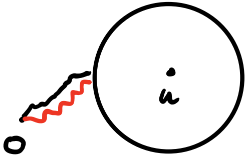
A.S.P. Q.I.P. (STRATEGY: CHEN, CROYDON, KUMAGAI 2012)

FOR $\mathbb{Q}$-A.E. α

$$\sup_{t\in[0,T]}\ \sup_{x\in\mathbb{Z}^d}\left| \mathcal{S}^{\alpha,RW}_{tN^2}\, G\left(\tfrac{x}{N}\right) - \mathcal{S}^{BM^{\Sigma}}_{t}\, G\left(\tfrac{x}{N}\right)\right| \longrightarrow 0$$

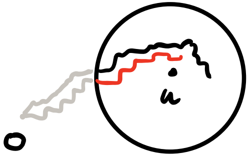
1° STEP) Q.I.P. FROM 0

2° STEP)



HITTING TIME CONVERGENCE

3° STEP)



CONVERGENCE AFTER HITTING

- EQUIC. OF $\mathcal{S}^{\alpha,RW}_t$
- HEAT KERNEL UPPER BOUNDS

TOWARDS

FRACTIONAL DIFFERENTIAL EQUATIONS

(ONGOING PROJECT WITH A. CHIARINI)

HEAVY TAILED ENVIRONMENT

$$\mathbb{Q}(\alpha_x \geq u) = c_1 u^{-\beta}(1 + o(1)) \quad \text{AS } u \to \infty$$

$$\boxed{\beta \in (0,1)}$$

HEAVY TAILED ENVIRONMENT

$$\mathbb{Q}(\alpha_x \geq u) = c_1 u^{-\beta}(1 + o(1)) \quad \text{AS } u \to \infty$$

$$\beta \in (0,1)$$

$(\alpha_x)_{x \in \mathbb{Z}^d}$ i.i.d.

$$\mathbb{Q}(\alpha_x \geq 1) = 1$$

Bouchaud's Trap Model

$X(t)$ R.W. on $\mathbb{Z}^d$ with Transition Rates

$$\alpha_x^{1-a}\,\alpha_y^{a} \qquad \left(a \in [0,1] \right)$$

BOUCHAUD'S TRAP MODEL

$X(t)$ R.W. ON $\mathbb{Z}^d$ WITH TRANSITION RATES

$$\alpha_x^{1-a}\,\alpha_y^{a} \qquad (a \in [0,1])$$

- OUR CASE : $a = 1$ $\qquad x \curvearrowright y \qquad \alpha_y$

Bouchaud's Trap Model

$X(t)$ R.W. on $\mathbb{Z}^d$ with transition rates

$$\alpha_x^{1-a}\,\alpha_y^{a} \qquad (a\in[0,1])$$

- our case: $a=1$
- $(\alpha_x)_{x\in\mathbb{Z}^d}$ is reversible
- $\dfrac{\eta(x)}{\alpha_x}$ is a duality function

The Fractional Kinetics Process

- $B(t)$ d-dim standard B.M.
- $V_\beta \perp\!\!\!\perp B$, β-stable subordinator

$$\mathbb{E}[e^{-\lambda V_\beta(t)}] = e^{-t\lambda^\beta}$$

- $V_\beta^{-1}(s) := \inf\{t : V_\beta(t) > s\}$

$$FK_\beta(s) := B(V_\beta^{-1}(s))$$

PROPERTIES OF $\left(FK_\beta(s)\right)_{s \geq 0}$

- NON-MARKOV
- γ-HÖLDER CONT., $\gamma < \beta/2$
- $FK_\beta(\cdot) \overset{\text{LAW}}{=} \lambda^{-\beta/2} FK_\beta(\lambda \cdot)$, $\lambda > 0$
- $p(t,x)$ SATISFIES

$$\frac{\partial^\beta}{\partial t^\beta} p(t,x) = \frac{1}{2} \Delta p(t,x) + \delta_0(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}$$

THEOREM (BARLOW, ČERNÝ 2011)

- $d \geq 3$

- $\mathbb{Q}(\alpha_x \geq u) = c_1 u^{-\beta}(1 + o(1))$ AS $u \to \infty$ $\quad \beta \in (0,1)$

$(\alpha_x)_{x \in \mathbb{Z}^d}$ i.i.d.

$\mathbb{Q}(\alpha_x \geq 1) = 1$

THEOREM (BARLOW, ČERNÝ 2011)

- $d \geq 3$
- $\mathbb{Q}(\alpha_x \geq u) = c_1 u^{-\beta}(1 + o(1))$ AS $u \to \infty$ $\quad \beta \in (0,1)$

  $(\alpha_x)_{x \in \mathbb{Z}^d}$ i.i.d.

  $\mathbb{Q}(\alpha_x \geq 1) = 1$

$$X_n(t) := \frac{1}{n} X(t n^{2/\beta})$$

THEOREM (BARLOW, ČERNÝ 2011)

- $d \geq 3$
- $\mathbb{Q}(\alpha_x \geq u) = c_1 u^{-\beta}(1 + o(1))$ AS $u \to \infty$ $\quad \beta \in (0,1)$

  $(\alpha_x)_{x \in \mathbb{Z}^d}$ i.i.d.

  $\mathbb{Q}(\alpha_x \geq 1) = 1$

$$X_n(t) := \frac{1}{n} X(t n^{2/\beta})$$

$\mathbb{Q}$-A.E. α

$$\boxed{X_n \Rightarrow c \, FK_\beta}$$

THEOREM (BARLOW, ČERNÝ 2011)

- $d \geqslant 3$

- $\mathbb{Q}(\alpha_x \geqslant u) = c_1 u^{-\beta}(1 + o(1))$ AS $u \to \infty$ $\quad \beta \in (0,1)$

  $(\alpha_x)_{x \in \mathbb{Z}^d}$ i.i.d.

  $\mathbb{Q}(\alpha_x \geqslant 1) = 1$

$$X_n(t) := \frac{1}{n} X(t n^{2/\beta})$$

THEN $\exists$ C S.T. $\mathbb{Q}$-A.E. α $(X_n)_n$ (STARTING AT 0) CONVERGES IN LAW TO $C\,FK_\beta$ in $D([0,\infty), \mathbb{R}^d)$

RANDOM TIME CHANGE

ANOTHER RW : VSRW(α)

$Y(t)$ $\quad \overset{\cdot}{x} \curvearrowright \overset{\cdot}{y}$ RATE $\omega_{xy} = \alpha_x \alpha_y$

RANDOM TIME CHANGE

ANOTHER RW : VSRW(α)

$Y(t)$ $\quad x \to y$ RATE $\omega_{xy} = \alpha_x \alpha_y$

$$S(t) := \int_0^t \alpha_{Y(s)} \, ds$$

- CONT.
- INCR.

RANDOM TIME CHANGE

ANOTHER RW : VSRW(α)

$Y(t)$ $\quad x \curvearrowright y$ $\quad$ RATE $\omega_{xy} = \alpha_x \alpha_y$

$$S(t) := \int_0^t \alpha_{Y(s)}\, ds$$

- CONT.
- INCR.

$$X(t) = Y(S^{-1}(t))$$

THEOREM (BARLOW, ČERNÝ 2011 + BARLOW, DEUSCHEL 2009)

- SAME ASSUMPTIONS
  - $d \geq 3$
  - $\mathbb{Q}(\alpha_x \geq u) = c_1 u^{-\beta}(1 + o(1))$ AS $u \to \infty$ $\quad \beta \in (0,1)$

    $(\alpha_x)_{x \in \mathbb{Z}^d}$ i.i.d.

    $\mathbb{Q}(\alpha_x \geq 1) = 1$

- $$Y_n(t) := \frac{Y(n^2 t)}{n}$$

  $$S_n(t) := n^{-2/\beta} S(n^2 t)$$

THEOREM (BARLOW, ČERNÝ 2011 + BARLOW, DEUSCHEL 2009)

- SAME ASSUMPTIONS
  - $d \geqslant 3$
  - $\mathbb{Q}(\alpha_x \geqslant u) = c_1 u^{-\beta}(1 + o(1))$ AS $u \to \infty$ $\quad \beta \in (0,1)$

    $(\alpha_x)_{x \in \mathbb{Z}^d}$ i.i.d.

    $\mathbb{Q}(\alpha_x \geqslant 1) = 1$

- $$Y_n(t) := \frac{Y(n^2 t)}{n}$$

  $$S_n(t) := n^{-2/\beta} S(n^2 t)$$

$$\boxed{\mathbb{Q}\text{-A.E. } \alpha \qquad (Y_n, S_n) \Rightarrow (C_1 B, C_2 V_\beta)}$$

CONSEQUENCES ON SEP(α)

- μ_N : $\mathbb{E}_{\mu_N}[\eta(x)] = \bar{\rho}\left(\frac{x}{N}\right)$

CONSEQUENCES ON SEP(α)

μ_N : $\mathbb{E}_{\mu_N}[\eta(x)] = \bar{\rho}\left(\frac{x}{N}\right)$

$$\Rightarrow \quad \mathbb{E}_{\mu_N}[\eta_{tN^{2/\beta}}(0)] = \mathcal{S}^{RW(\alpha)}_{tN^{2/\beta}}\bar{\rho}(0)$$

CONSEQUENCES ON SEP(α)

- μ_N : $\int \eta(x)\, d\mu_N = \bar{\rho}(\frac{x}{N})$

$$\Rightarrow \quad \mathbb{E}_{\mu_N}[\eta_{tN^{2/\beta}}(0)] = \mathcal{S}^{RW(\alpha)}_{tN^{2/\beta}} \bar{\rho}(0)$$

$$\Big\downarrow{\scriptstyle N \to \infty}$$

$$\mathbb{E}_0^{FK_\beta}[\bar{\rho}(FK_\beta(t))]$$

CONSEQUENCES ON SEP(α)

- μ_N : $\int \eta(x)\, d\mu_N = \bar{\rho}\left(\frac{x}{N}\right)$

$$\Rightarrow \quad \mathbb{E}_{\mu_N}\left[\eta_{tN^{2/\beta}}(0)\right] = \mathcal{S}^{RW(\alpha)}_{tN^{2/\beta}}\, \bar{\rho}(0)$$

$$\Big\downarrow N \to \infty$$

$$\mathbb{E}_0^{FK_\beta}\left[\bar{\rho}(FK_\beta(t))\right]$$

$$\Rightarrow \quad \text{HDL} = \begin{cases} \dfrac{\partial^\beta}{\partial t^\beta}\rho(t,x) = \dfrac{1}{2}\Delta\rho(t,x) + \delta_0(x)\,\dfrac{t^{-\beta}}{\Gamma(1-\beta)} \\ \rho(0,x) = \bar{\rho}(x) \end{cases}$$

Final Questions

- A.S.P. Q.I.P. for VSRW Y_m ?

  ✓ PHI + heat kernel upper bounds

- A.S.P. Q.I.P. for S_m ? ✓

- L^1 conv. expectations ? $\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\left|\mathcal{J}^{\alpha,RW}_{tN^{2/\beta}}G\left(\tfrac{x}{N}\right)-\mathbb{E}^{FK_\beta}_{\frac{x}{N}}\left[G\left(FK_\beta(t)\right)\right]\right|\longrightarrow 0$

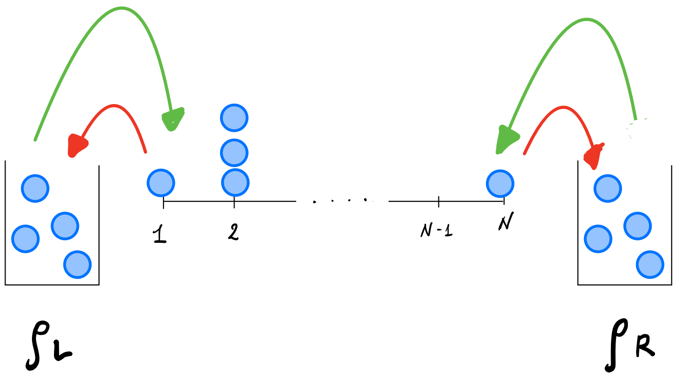
- Heat kernel estimates for $X_m(t)$?

  $? \quad p_t(x,y) \simeq t^{-d/2\,\beta}\left(\frac{d(x,y)^2}{t^\beta}\right)^{1-d/2} \quad ?$

THE END

= PARTICLE (RANDOM WALK)

$$\eta_t = \left(\eta_t(x)\right)_{x \in \{1,\dots,N\}}$$



$\rho_L \neq \rho_R$ **OUT OF EQUILIBRIUM**

i) μ_{STAT} $\exists!$

ii) $\mathbb{E}_{\mu_{STAT}}\left[\left(\eta(x) - \mathbb{E}_{\mu_{STAT}}[\eta(x)]\right)\left(\eta(y) - \mathbb{E}_{\mu_{STAT}}[\eta(y)]\right)\right]$

iii) FICK LAW

EXAMPLE: SIMPLE SYMMETRIC EXCLUSION PROCESS (SEP)

at Rate $(1-\eta(y))$

$$\boxed{A g(x) = \sum_{y:|y-x|=1} (g(y)-g(x))}$$

DUALITY FOR SEP

$$\mathcal{L}\left(\eta(x)\right)[\eta] = A\left(\eta(\cdot)\right)[x]$$

TO BE PRECISE ... MILD SOLUTIONS

$$\begin{cases} d\left(\frac{\eta_t}{\alpha_\cdot}\right)(x) = A^\alpha\left(\frac{\eta_{t^-}}{\alpha_\cdot}\right)(x)\,dt + dM_t\left(x, \frac{\eta}{\alpha}\right) \\ \left(\frac{\eta_0}{\alpha}\right)(x) = \frac{\bar{\eta}(x)}{\alpha_x} \qquad \forall t \geq 0,\ \forall x \in \mathbb{Z}^d \end{cases}$$

$$\left(\frac{\eta_t}{\alpha}\right)(x) = \mathcal{S}_t^{\alpha, RW}\left(\frac{\bar{\eta}}{\alpha}\right)(x) + \int_0^t \mathcal{S}_{t-s}^{\alpha, RW}\, dM_t\left(x, \frac{\eta_{t^-}}{\alpha}\right)$$

THEOREM (BARLOW, ČERNÝ 2011 + BARLOW, DEUSCHEL 2009)

- SAME ASSUMPTIONS
  - $d \geq 3$
  - $\mathbb{Q}(\alpha_x \geq u) = c_1 u^{-\beta}(1+o(1))$ AS $u \to \infty$ $\quad \beta \in (0,1)$

    $(\alpha_x)_{x \in \mathbb{Z}^d}$ i.i.d.

    $\mathbb{Q}(\alpha_x \geq 1) = 1$

- $$Y_n(t) := \frac{Y(n^2 t)}{n}$$

  $$S_n(t) := n^{-2/\beta} S(n^2 t)$$

THEN $\exists$ $C_Y, C_S \in (0,\infty)$ S.T. $\mathbb{Q}$-A.E. α THE JOINT DIST. OF (S_N, Y_N) CONV. TO THE DIST. OF $(C_S V_\beta, C_Y B)$ WEAKLY