Recent progress in relativistic collisionless kinetic theory

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Summary

(1) A crash course in General Relativity with an introduction to the Einstein-Vlasov system.

(2) Global existence theorems for the Einstein-Vlasov system with small data.

(3) The analysis of Vlasov fields around black holes.

Newton to Einstein

- Newton's (2nd) law:

$$m\ \mathbf{a} = m\ \frac{d^2\,\mathbf{x}}{dt^2} = \sum \mathbf{F}, \quad (NL)$$

- In case of free fall, $\mathbf{F} = -\nabla\phi$, where ϕ is the gravitational potential, solution to Poisson equation

$$\Delta\phi = 4\pi G\rho, \quad ($$  $$)$$

- ρ is the mass density, for a distribution of particles (with $m = 1$)

$$\rho = \int_{v\in\mathbb{R}^3} f dv$$

- From (NL), we get the Vlasov eq:

$$\partial_t f + v.\nabla_x\phi - \nabla_x\phi.\nabla_v f = 0. \quad (V)$$

To get the analogue of the Vlasov-Poisson system in General Relativity, we need

- The replacement of Newton's 2nd law (NL), describing the behavior of free falling particles.
- The replacement of Poisson equation (), describing how the distribution of matter influences gravity.
- The replacement of ρ: what macroscopic quantities describe mass or energy of a distribution of particles.

A crash course in General Relativity

(with low-tech geometrical prerequisites)

- Analysis takes place on a $4d$ manifold, the *spacetime* manifold $\mathcal{M}$.
- Here, $\mathcal{M} = \mathbb{R}^4$, parametrized by $(x^\alpha)_{0\leq\alpha\leq 3} = (x^0 = t, x^1, x^2, x^3)$.
- Gravity is described via a Lorentzian metric g: locally

$$g : (x^\alpha) \to g_{\beta\delta}(x^\alpha) \in M_4(\mathbb{R}),$$

where

  - $g_{\beta\delta} = g_{\beta\delta}$ is symmetric,
  - at all points, $g_{\beta\delta}$ has 1 ev < 0, 3 ev > 0.

- Standard example is Minkowski space $g_{\beta\delta} = \mathrm{diag}(-1, 1, 1, 1)$.

Motion of particles in General Relativity

- a particle moves along curve $\gamma : s \in I \to (x^\alpha(s)) \in \mathcal{M}$ with

$$\begin{aligned} g(\dot\gamma,\dot\gamma) &< 0, \quad \text{massive particle,} \\ g(\dot\gamma,\dot\gamma) &= 0, \quad \text{massless particle (photon).} \end{aligned}$$

- Free fall is described by the geodesic equation in the geometry of g:

$$\frac{d^2x^\alpha}{ds^2} + \sum_{\beta,\delta} \frac{dx^\beta}{ds}\frac{dx^\delta}{ds}\Gamma^\alpha_{\beta\delta} = 0, \quad (Geo)$$

where $\Gamma^\alpha_{\beta\delta} = \frac{1}{2}\sum_\rho (g^{-1})^{\alpha\rho}\left(g_{\rho\beta,\delta} + g_{\rho\delta,\beta} - g_{\beta\delta,\rho}\right) \simeq g^{-1}\partial g$

- Schematically

$$(Geo): \ \frac{d^2x}{ds^2} = -\frac{dx}{ds}\frac{dx}{ds}g^{-1}\partial g, \quad (NL): \ \frac{d^2x}{ds^2} = -\partial\phi.$$

The Einstein equations I

- The metric g is schematically the replacement for ϕ.
- The replacement of Poisson equation are the *Einstein equations*

$$\mathrm{Ric}(g) - \frac{1}{2}S(g)g = \mathrm{T}, \quad (Ee)$$

where Ric is the Ricci tensor, S the scalar curvature,

- T, the energy-momentum tensor, is the replacement for ρ and depends on a choice of matter models.
- The LHS of (Ee) can be viewed as a second order, non-linear, differential operator applied to g.

The Einstein equations II

- Similar to Maxwell eq, we special gauges (think Lorentz or Coulomb gauge) to write (Ee) as a system of well-posed pdes.
- In the so-called wave gauge, (Ee) becomes

$$\Box_g g_{\alpha\beta} = Q_{\alpha\beta}(\partial g, \partial g) + T_{\alpha\beta}$$

- Here $\Box_g \psi = \frac{1}{\sqrt{|\det g|}} \sum_{\alpha,\beta} \partial_\alpha \left((g^{-1})^{\alpha\beta} \sqrt{|\det g|} \partial_\beta \psi \right)$ is the *wave operator* in the metric g.

Some facts concerning the Einstein equations

- The Einstein equations with zero source term $T \equiv 0$ reduces to vacuum Einstein equations

$$Ric(g) = 0$$

which admits plenty of non-trivial solutions, in contrast to $\Delta\phi = 0$.

- The Einstein equations, in vacuum or with suitable matter, admits a well-posed initial value problem (Choquet-Bruhat 1952).
- Einstein equations are quasilinear wave equations $\rightarrow$ requires high regularity data, at least $g \in H^2$ (Klainerman-Rodnianski-Szeftel (2015)), in contrast to $\Delta\phi = \rho$.
- In fact, for most results, we need much more, and, in particular, need to commute the equations.

The Einstein-Vlasov system I

The Einstein-Vlasov system is composed of

- The Einstein equations:

$$Ric(g) - \frac{1}{2} g S(g) = T[f],$$

  where $T[f]$ is the energy-momentum tensor of a distribution of particles $f = f(x^\alpha, v^\alpha)$,

- The Vlasov equation verified by f

$$\sum_\alpha v^\alpha \partial_{x^\alpha} f - \sum_{\alpha,\beta} v^\alpha v^\beta \Gamma^\delta_{\alpha\beta} \partial_{v^\delta} f = 0,$$

- and the definition of $T[f]$:

$$T_{\alpha\beta}[f] = \int_v v_\alpha v_\beta f d\mu_v.$$

The Einstein-Vlasov system II

- The function f is defined in phase space: here the tangent or cotangent bundle of $\mathcal{M}$.
- In $\mathbb{R}^4$, this means that $f = f(x^\alpha, v^\alpha)$
- Fixing a mass $m = 1$ of the particles is then equivalent to having the support of f restricted to points $g_x(v, v) = -1$.
- This restricts the domain of integration in v and fix the measure $d\mu_v$ in the definition of T.
- In Minkowski space

$$\begin{aligned} f &= f(t, x^1, x^2, x^3, v^1, v^2, v^3), \quad v^0 = \sqrt{1+|v|^2}, \\ T_{\alpha\beta}[f] &= \int_{v\in\mathbb{R}^3} v_\alpha v_\beta \, f \, \frac{dv}{\sqrt{1+|v|^2}}. \end{aligned}$$

Small data global existence in the vacuum case: the stability of the Minkowski space

- The vacuum equations can be written as $\Box_g g = Q(\partial g, \partial g)$.
- The trivial solution is Minkowski space $g = \eta = \mathrm{diag}(-1, 1, 1, 1)$.
- Small data problem: for initial data close to the data of η, prove that there exists a global solution g, and that this converges asymptotically to η.
- First addressed by Christodoulou-Klainerman (93), with many subsequent proofs, in particular Linbldad-Rodnianski (2010).
- Many difficulties (data geometric, constraints, gauge, tensorial equations, etc..)

Main difficulty: slow decay of linear waves in 3d

- If ψ solves $\Box\psi := (-\partial_t^2 + \Delta)\psi = 0$, in 3d with compact smooth data, then

$$||\partial\psi(t,\,.\,)||_{L^\infty_x} \lesssim \frac{1}{t}.$$

- Consider a model problem of the form

$$\Box\psi = (\partial\psi)^2$$

and assume the linear decay rate.

- Let $E(t) = \int_{x\in\mathbb{R}^3} |\partial_t\psi(t,x)|^2 + |\nabla_x\psi(t,x)|^2 dx$, then,

$$E(t) \lesssim E(0)\exp\left(\int_0^t ||\partial\psi(s,\,.\,)||_{L^\infty_x} ds\right)$$

which will diverges logarithmically in t.

Null forms I

- For general model problems $\Box\psi = Q(\partial\psi, \partial\psi)$, this is the best one can do (blow up can occur, John (81)).
- However, global existence holds for special Q such as

$$Q(\partial\psi, \partial\psi) = -|\partial_t\psi|^2 + |\nabla_x\psi|^2,$$

called *null forms* (Christodoulou (86), Klainerman (86))

- Improve decay estimates can be obtained for such null forms because they always one derivatives tangential to the light cone, such as $\partial_t + \partial_r$, for which, linear estimate is improved

$$||(\partial_t + \partial_r)\psi(t, \,.\,)||_{L^\infty} \lesssim \frac{1}{t^2}.$$

Null forms II

- For the Einstein equations, the non-linearity depends on the formulation (choice of gauge).
- In the wave gauge, non-linear terms are not all null forms !!
- CK used a more geometric formulation, LR showed that one can allow growth and exploit a weaker, triangular structure:

$$\begin{aligned} \Box u &= 0, \\ \Box v &= (\partial_t u)^2. \end{aligned}$$

Theorem (Fajman-Joudioux-J.S. and Taylor-Lindblad, 2017)

Minkowski space is stable as the trivial solution to the Einstein-Vlasov system: for initial data close to Minkowski and small, localized data for f*, (g,f) exists globally,* g *converges to* η *and* $T[f]$ *goes to* 0*.*

- Previous result in spherical symmetry to Rendall-Rein (1992)

Remarks

- Small data global existence for Vlasov-Poisson in $\mathbb{R}^3$: Bardos-Degond (85).
- Relies on decay for $\rho(f) = \int_v f dv$ and elliptic estimates for $\nabla\phi$.
- No need to obtain good, sharp estimates after commutation.
- In constract, Einstein equations require commutation, and one need sharp decay estimates for derivatives.
- Sharp decay for VP : (Hwang-Rendall-Velázquez (2011), J.S (2016), Ionescu-Pausader-Wang-Widmayer (2020)):

$$|\rho(f)| \lesssim \frac{1}{t^3}, \quad |\rho(\partial_x f)| \lesssim \frac{1}{t^4}.$$

- One can also consider the Einstein-Vlasov system for massless particles
- In this case, f is supported on (x, v) with $g_x(v, v) = 0$.
- At the linear level, the transport operator is then

$$\left(\partial_t + \frac{v}{|v|}.\nabla_x\right) f.$$

- For compact initial data, support of the solution stays in a strip near $t = |x|$ leading to semi-global problem.
- Stability of Minkowski space for massless particles with compact data: Dafermos (2006) , Taylor (2016).
- The non-compact case (optimal in v): Fajman-Joudioux-Thaller-Bigorgne, J.S. (2020).

Commutation vector fields for relativistic operators I

- Key ingredient of the proofs: how to commute both the Einstein equations and the Vlasov equations.
- Start with commuting the linear wave operator $\Box\psi := [-\partial_t^2 + \Delta]\psi$.
- Due to the symmetry of the Minkowski space, $[\Box, Z] = 0$ with Z any of the vector fields:
  - Translations: ∂_t, ∂_x.
  - Rotations: $x^i\partial_{x^j} - x^j\partial_{x^i}$.
  - Hyperbolic rotations: $t\partial_{x^i} + x^i\partial_t$.
  - Scaling $S := t\partial_t + x^i\partial_{x^i}$, for which $[\Box, S] = 2\Box$.
- Thus, estimates we can prove for solutions to the wave equation ψ, we can prove for $Z\psi$, $Z^2\psi$ etc..
- Importantly, these vector fields have weights in t and x.
- Klainerman-Sobolev inequality exploiting the algebra of commuting vector fields.

$$|\psi(t,x)| \lesssim \frac{1}{(1+t+|x|)(1+|t-|x|)^{1/2}} \sum_{|\alpha|\leq 3} ||Z^\alpha \psi(t,\ .)||_{L^2_x}$$

Commutation vector fields for relativistic operators II

- Consider now the free transport operator (say massless) $Tf = \partial_t + \frac{v}{|v|}\partial_x f$.
- To any of the previous Z vector fields, defined in physical space, correspond a *lifted* vector field, $\widehat{Z}$ in phase space, that commutes with $[T, \widehat{Z}]$:
  - Lifted translations: $\widehat{\partial_{t,x}} = \partial_x$.
  - Lifted rotations: $x^i\partial_{x^j} - x^j\partial_{x^i} + v^i\partial_{v^j} - v^j\partial_{v^i}$.
  - Hyperbolic rotations: $t\partial_{x^i} + x^i\partial_t + |v|\partial_{v^i}$.
  - (non-lifted) Scaling $S := t\partial_t + x^i\partial_{x^i}$, $[T, S] = T$.
- Importantly, these vector fields have weights in t and x.
- Klainerman-Sobolev inequality for Vlasov fields [FJS].

$$\int_v |f|(t,x,v)|v|dv \lesssim \frac{1}{(1+t+|x|)^2(1+|t-|x|)} \sum_{|\alpha|\leq 3} ||\widehat{Z}^\alpha f(t,\ .,\ .)||_{L^1_{x,v}}$$

Extra difficulties and ingredients

- For the non-linear transport operator, schematically

$$[T + \partial g \partial_v, \widehat{Z}] = Z \partial g . \partial_v .$$

- But $\partial_v f$ grows linearly in t for linear solutions.
- For massive particles, our proof relies on modified vector fields $Y := \widehat{Z} + \Phi . \partial_{t,x}$, for coefficients Φ that verifies themselves a non-linear transport equation.
- This allows to kill the worse growing term, but the vector fields depend now on the solutions.
- Not needed in massless case.
- In both cases, we also use null and weak null (triangular) structure in the Vlasov equation:
  - Ex:

$$\nabla_x g . \nabla_v = \frac{Z(g)}{t} \cdot \partial_v - \frac{x}{t} \frac{\partial_t g}{\sqrt{1+|v|^2}} \widehat{Z} + \frac{\partial_t g}{\sqrt{1+|v|^2}} \left(S + \frac{|x|^2 - t^2}{t} \partial_t \right) .$$

Black Holes

- The basic black hole solution is the so-called Schwarzschild black hole of mass $M > 0$, whose metric, outside from the black hole is given by

$$g_{Sch} = -\left(1 - \frac{2M}{r}\right) dt \otimes dt + \left(1 - \frac{2M}{r}\right)^{-1} dr \otimes dr + r^2 \sigma_{\mathbb{S}^2}.$$

- It is a spherically-symmetric solution to the vacuum Einstein equations,
  - as $r \to +\infty$, the metric converge to that of Minkowski,
  - The metric can be extended through $r = 2M$, with $r > 2M$ being the exterior region.
  - Nonlinear stability of the exterior of Schwarzschild and more generally of the *Kerr* spacetime: important problem in GR (Dafermos-Holzegel-Rodnianski, Klainerman-Szeftel-Giorgi)

Massless Vlasov equation outside Schwarzschild

Consider the massless Vlasov equation in the exterior of the Schwarzschild solution:

$$\begin{aligned} 0 &= -\frac{v_t}{1-\frac{2M}{r}}\partial_t f + \frac{v_{r^*}}{1-\frac{2M}{r}}\partial_{r^*} f + \frac{v_\theta}{r^2}\partial_\theta f + \frac{v_\varphi}{r^2\sin^2(\theta)}\partial_\varphi f \\ &\quad + \frac{r-3M}{r^4}|\not{v}|^2\partial_{v_{r^*}} f + \frac{\cot(\theta)}{r^2\sin^2(\theta)} v_\varphi^2 \partial_{v_\theta} f, \\ r^* &:= r + 2M\log(r-2M) - 3M - 2M\log M, \end{aligned}$$

- Superpolynomial decay of velocity averages Bigorgne, 2020, non-compact data.
- Exponential decay: Velozo Ruiz, compact data
- Nonlinear Stability of Schwarzshild for the Einstein-massless-Vlasov system in spherical-symmetric, Velozo Ruiz.
- Main difficulty: trapping, turning points $v_{r^*} = 0$, lack of good commutation vector fields.
- Open problem 1: prove strong quantitative estimates for derivatives.
- Open problem 2: prove optimal decay estimates with non-compact support (sharp in terms of the initial norms).

Massive Vlasov equation outside Schwarzschild

- Geodesic equation in Schwarzschild is completely integrable.
- Radial motion is equivalent to that of a particle in 1d with effective potential $E_\ell^{Sch} = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell}{r^2}\right)$

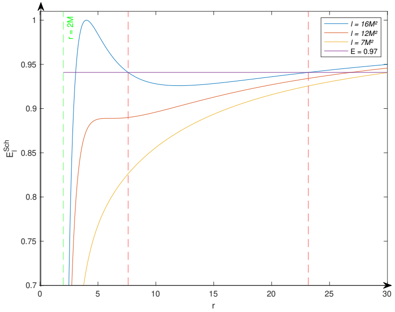

Figure: Shape of the effective potential energy E_ℓ^{Sch} for three cases of ℓ, with $M = 1$, $E = 0.97$, credit F.E. Jabiri.

Massive Vlasov fields outside Schwarzschild

Results:

- Existence of steady states for the EV system in spherical-symmetry outside black holes: Rein, Jabiri 2020
- Linear stability of such steady states: Günther-Rein-Straub, 2022
- Axisymmetric steady states by deforming Kerr: Jabiri 2022.
- Mixing for trapped Vlasov field (Rioseco-Sarbach).
- Linear decay estimates for non-trapped Vlasov field (Velozo-Ruiz)

Open problems:

- Prove quantitative mixing estimates for trapped Vlasov field.
- Classify all possibly spherically-symmetric steady states near Schwarzschild (Jean's theorem)
- Stability in the axisymmetric case