Algebraic function based Banach space valued ordinary and fractional neural network approximations

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Abstract

Here we research the univariate quantitative approximation, ordinary and fractional, of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. These approximations are derived by establishing Jackson type inequalities invovling the modulus of continuity of the engaged function or its Banach space valued high order derivative of fractional derivatives. Our operators are defined by using a density function generated by an algebraic sigmoid function. The approximations are pointwise and of the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer.

2020 AMS Mathematics Subject Classification : 26A33, 41A17, 41A25, 41A30, 46B25.
Keywords and Phrases: algebraic sigmoid function, Banach space valued neural network approximation, Banach space valued quasi-interpolation operator, modulus of continuity, Banach space valued Caputo fractional derivative, Banach space valued fractional approximation, iterated fractional approximation.

1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rated by very specifically defined neural network operators of Cardaliagnet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped "and "squashing "functions are assumed to be of compact suport. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

The author inspired by [14], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3], [4], [5], [6], [7], by treating both the univariate and multivariate cases. He did also the corresponding fractional cases [8], [9], [13].

The author here performs algebraic sigmoidal based neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with valued to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related X-valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X-valued high order derivative, or X-valued fractional derivatives and given by very tight Jackson type inequalities. Iterated fractional approximation is also included.

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we establish important properties of the basic density function defining our operators which is induced by algebraic sigmoidal function.

Feed-forward X-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{=0}^{n} c_j \sigma \left(\langle a_j \cdot x \rangle + b_j \right), \quad x \in \mathbb{R}^s, \ \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. About neural networks in general read [15], [17], [19]. See also [9] for a complete study of real valued approximation by neural network operators.

2 Basics

We consider the generator algebraic function

$$\varphi(x) = \frac{x}{\sqrt[2m]{1 + x^{2m}}}, \quad m \in \mathbb{N}, x \in \mathbb{R}, \tag{1}$$

which is a sigmoidal type of function and is a strictly increasing function.

We see that $\varphi(-x) = -\varphi(x)$ with $\varphi(0) = 0$. We get that

$$\varphi'(x) = \frac{1}{\left(1+x^{2m}\right)^{\frac{2m+1}{2m}}} > 0, \quad \forall\, x \in \mathbb{R}, \tag{2}$$

proving φ as strictly increasing over $\mathbb{R}, \varphi'(x) = \varphi'(-x)$. We easily find that $\lim_{x\to+\infty} \varphi(x) = 1$, $\varphi(+\infty) = 1$, and $\lim_{x\to-\infty} \varphi(x) = -1$, $\varphi(-\infty) = -1$.

We consider the activation function

$$\Phi(x) = \frac{1}{4}\left[\varphi(x+1) - \varphi(x-1)\right]. \tag{3}$$

Clearly it is $\Phi(x) = \Phi(-x)$, $\forall\, x \in \mathbb{R}$, so that Φ is an even function and symmetric with respect to the y-axis.

Since $x+1 > x-1$, we have $\varphi(x+1) > \varphi(x-1)$ and $\Phi(x) > 0$, $\forall\, x \in \mathbb{R}$.

Also it is

$$\Phi(0) = \frac{1}{2\sqrt[2m]{2}}. \tag{4}$$

We observe that

$$\Phi'(x) = \frac{1}{4}\left(\varphi'(x+1) - \varphi'(x-1)\right) =$$

$$\frac{1}{4}\left(\frac{1}{\left(1+(x+1)^{2m}\right)^{\frac{2m+1}{2m}}} - \frac{1}{\left(1+(x-1)^{2m}\right)^{\frac{2m+1}{2m}}}\right), \quad \forall\, x \in \mathbb{R}. \tag{5}$$

Let now $x > 0$, then $x > -x$ and $(x+1)^2 > (x-1)^2 \geq 0$, implying $(x+1)^{2m} > (x-1)^{2m} \geq 0$, $m \in \mathbb{N}$, and $1 + (x+1)^{2m} > 1 + (x-1)^{2m} > 0$. Consequently it holds

$$\frac{1}{\left(1+(x-1)^{2m}\right)^{\frac{2m+1}{2m}}} > \frac{1}{\left(1+(x+1)^{2m}\right)^{\frac{2m+1}{2m}}}, \tag{6}$$

proving $\Phi'(x) < 0$ for $x > 0$.

That is Φ is strictly decreasing over $(0, +\infty)$.

Clearly, Φ is strictly increasing over $(-\infty, 0)$ and $\Phi'(0) = 0$.

Furthermore we obtain that

$$\lim_{x\to+\infty} \Phi(x) = \frac{1}{4}\left[\varphi(+\infty) - \varphi(+\infty)\right] = 0, \tag{7}$$

and

$$\lim_{x\to-\infty} \Phi(x) = \frac{1}{4}\left[\varphi(-\infty) - \varphi(-\infty)\right] = 0. \tag{8}$$

That is the x-axis is the horizontal asymptote of Φ.

Conclusion, Φ is a bell symmetric function with maximum

$$\Phi(0) = \frac{1}{2\sqrt[2m]{2}}, \quad m \in \mathbb{N}. \tag{9}$$

We need

Theorem 1 *We have that*

$$\sum_{i=-\infty}^{\infty} \Phi(x-i) = 1, \ \ \forall\, x \in \mathbb{R}. \tag{10}$$

Proof. We observe that

$$\sum_{i=-\infty}^{\infty} \left(\varphi(x-i) - \varphi(x-1-i)\right) =$$

$$\sum_{i=0}^{\infty} \left(\varphi(x-i) - \varphi(x-1-i)\right) + \sum_{i=-\infty}^{-1} \left(\varphi(x-i) - \varphi(x-1-i)\right).$$

Furthermore ($\lambda \in \mathbb{Z}^+$)

$$\sum_{i=0}^{\infty} \left(\varphi(x-i) - \varphi(x-1-i)\right) = \tag{11}$$

$$\lim_{\lambda\to\infty} \sum_{i=0}^{\lambda} \left(\varphi(x-i) - \varphi(x-1-i)\right) \quad \text{(telescoping sum)}$$
$$= \lim_{\lambda\to\infty} \left(\varphi(x) - \varphi(x-(\lambda+1))\right) = 1 + \varphi(x).$$

Similarly, it holds

$$\sum_{i=-\infty}^{-1} \left(\varphi(x-i) - \varphi(x-1-i)\right) = \lim_{\lambda\to\infty} \sum_{i=-\lambda}^{-1} \left(\varphi(x-i) - \varphi(x-1-i)\right) \tag{12}$$
$$= \lim_{\lambda\to\infty} \left(\varphi(x+\lambda) - \varphi(x)\right) = 1 - \varphi(x).$$

Therefore we derive

$$\sum_{i=-\infty}^{\infty} \left(\varphi(x-i) - \varphi(x-1-i)\right) = 2, \ \ \forall\, x \in \mathbb{R}, \tag{13}$$

and

$$\sum_{i=-\infty}^{\infty} \left(\varphi(x+1-i) - \varphi(x-i)\right) = 2, \ \ \forall\, x \in \mathbb{R}. \tag{14}$$

Adding (13) and (14) we find

$$\sum_{i=-\infty}^{\infty} \left(\varphi(x+1-i) - \varphi(x-1-i)\right) = 4, \ \ \forall\, x \in \mathbb{R}. \tag{15}$$

Clearly, then

$$\Phi(x-i) = \frac{1}{4}\left[\varphi(x+1-i) - \varphi(x-1-i)\right],$$

proving (10). ■

We make

Remark 2 *Because* Φ *is even it holds*

$$\sum_{i=-\infty}^{\infty} \Phi(i-x) = 1, \ \ \forall \ x \in \mathbb{R}.$$

Hence

$$\sum_{i=-\infty}^{\infty} \Phi(i+x) = 1, \ \ \forall \ x \in \mathbb{R},$$

and

$$\sum_{i=-\infty}^{\infty} \Phi(x+i) = 1, \ \ \forall \ x \in \mathbb{R}. \tag{16}$$

Theorem 3 *It holds*

$$\int_{-\infty}^{\infty} \Phi(x)\, dx = 1. \tag{17}$$

Proof. We observe that

$$\int_{-\infty}^{\infty} \Phi(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{j}^{j+1} \Phi(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{0}^{1} \Phi(x+j)\, dx = \tag{18}$$

$$\int_0^1 \left(\sum_{j=-\infty}^{\infty} \Phi(x+j)\, dx \right) = \int_0^1 1 dx = 1.$$

So $\Phi(x)$ is a density function on $\mathbb{R}$. ■

We need

Theorem 4 *Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *It holds*

$$\sum_{\begin{cases} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \Phi(nx-k) < \frac{1}{4m\left(n^{1-\alpha}-2\right)^{2m}}, \quad m \in \mathbb{N}. \tag{19}$$

Proof. We have that

$$\Phi(x) = \frac{1}{4}\left[\varphi(x+1) - \varphi(x-1)\right], \ \ \forall \ x \in \mathbb{R}.$$

Let $x \geq 1$. That is $0 \leq x-1 < x+1$. Applying the mean value theorem we get

$$\Phi(x) = \frac{1}{4} \cdot 2 \cdot \varphi'(\xi) = \frac{1}{2\left(1+\xi^{2m}\right)^{\frac{2m+1}{2m}}} > 0, \tag{20}$$

where $0 \leq x-1 < \xi < x+1$.

Then,

$$(x-1)^2 < \xi^2 < (x+1)^2$$

$$(x-1)^{2m} < \xi^{2m} < (x+1)^{2m}$$
$$1+(x-1)^{2m} < 1+\xi^{2m} < 1+(x+1)^{2m}$$
$$\left(1+(x-1)^{2m}\right)^{\frac{2m+1}{2m}} < \left(1+\xi^{2m}\right)^{\frac{2m+1}{2m}} < \left(1+(x+1)^{2m}\right)^{\frac{2m+1}{2m}}$$
$$\frac{1}{2\left(1+\xi^{2m}\right)^{\frac{2m+1}{2m}}} < \frac{1}{2\left(1+(x-1)^{2m}\right)^{\frac{2m+1}{2m}}}. \tag{21}$$

Hence

$$\Phi(x) < \frac{1}{2\left(1+(x-1)^{2m}\right)^{\frac{2m+1}{2m}}}, \quad \forall\ x \geq 1. \tag{22}$$

Thus, we have

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \Phi(nx-k) = \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \Phi(|nx-k|) <$$

$$\frac{1}{2} \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} \frac{1}{\left(1+(|nx-k|-1)^{2m}\right)^{\frac{2m+1}{2m}}} \leq \tag{23}$$

$$\frac{1}{2}\int_{(n^{1-\alpha}-1)}^{\infty} \frac{1}{\left(1+(x-1)^{2m}\right)^{\frac{2m+1}{2m}}} dx = \frac{1}{2}\int_{n^{1-\alpha}-2}^{\infty} \frac{1}{\left(1+z^{2m}\right)^{\frac{2m+1}{2m}}} dz =: (*).$$

We see that

$$1+z^{2m} > z^{2m}$$
$$\left(1+z^{2m}\right)^{\frac{2m+1}{2m}} > z^{2m+1}$$
$$\frac{1}{z^{2m+1}} > \frac{1}{\left(1+z^{2m}\right)^{\frac{2m+1}{2m}}}. \tag{24}$$

Therefore it holds

$$(*) < \frac{1}{2}\int_{n^{1-\alpha}-2}^{\infty} \frac{1}{z^{2m+1}} dz = \frac{1}{2}\int_{n^{1-\alpha}-2}^{\infty} z^{-(2m+1)} dz =$$

$$\frac{1}{2}\left(\frac{z^{-(2m+1)+1}}{-(2m+1)+1}\right)\Bigg|_{n^{1-\alpha}-2}^{\infty} = \frac{1}{2}\left(-\frac{z^{-2m}}{2m}\right)\Bigg|_{n^{1-\alpha}-2}^{\infty} = \tag{25}$$

$$\frac{z^{-2m}}{4m}\Bigg|_{\infty}^{n^{1-\alpha}-2} = \frac{\left(n^{1-\alpha}-2\right)^{-2m}}{4m} - \frac{(\infty)^{-2m}}{4m} = \frac{\left(n^{1-\alpha}-2\right)^{-2m}}{4m},$$

proving (19). ■

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

Theorem 5 *Let $[a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} < 2\left(\sqrt[2m]{1+4^m}\right), \tag{26}$$

$\forall\ x \in [a,b]$, $m \in \mathbb{N}$.

Proof. Let $x \in [a,b]$. We see that

$$1 = \sum_{k=-\infty}^{\infty} \Phi(nx-k) > \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) =$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(|nx-k|) > \Phi(|nx-k_0|), \tag{27}$$

$\forall\ k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$.

We can choose $k_0 \in [\lceil na \rceil, \lfloor nb \rfloor] \cap \mathbb{Z}$ such that $|nx-k_0| < 1$.

Therefore we get that

$$\Phi(|nx-k_0|) > \Phi(1) = \frac{1}{4}\left(\frac{2}{\sqrt[2m]{1+2^{2m}}}\right) = \frac{1}{2\ \sqrt[2m]{1+2^{2m}}}, \tag{28}$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(|nx-k|) > \frac{1}{2\ \sqrt[2m]{1+2^{2m}}}. \tag{29}$$

That is

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(|nx-k|)} < 2\ \sqrt[2m]{1+4^m}, \tag{30}$$

proving the claim. ■

We make

Remark 6 *We also notice that*

$$1 - \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nb-k) = \sum_{k=-\infty}^{\lceil na \rceil - 1} \Phi(nb-k) + \sum_{k=\lfloor nb \rfloor + 1}^{\infty} \Phi(nb-k)$$

$$> \Phi(nb - \lfloor nb \rfloor - 1) \tag{31}$$

(call $\varepsilon := nb - \lfloor nb \rfloor$, $0 \leq \varepsilon < 1$)

$$= \Phi(\varepsilon - 1) = \Phi(1-\varepsilon) \geq \Phi(1) > 0.$$

Therefore

$$\lim_{n\to\to\infty}\left(1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nb-k\right)\right)>0. \tag{32}$$

Similarly, it holds

$$1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(na-k\right)=\sum_{k=-\infty}^{\lceil na\rceil-1}\Phi\left(na-k\right)+\sum_{k=\lfloor nb\rfloor+1}^{\infty}\Phi\left(na-k\right)$$

$$>\Phi\left(na-\lceil na\rceil+1\right) \tag{33}$$

(call $\eta:=\lceil na\rceil-na$, $0\leq\eta<1$*)*

$$=\Phi\left(1-\eta\right)\geq\Phi\left(1\right)>0.$$

Therefore again

$$\lim_{n\to\infty}\left(1-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(na-k\right)\right)>0. \tag{34}$$

Here we find that

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)\neq1,\quad \textit{for at least some } x\in\left[a,b\right]. \tag{35}$$

Note 7 *Let* $[a,b]\subset\mathbb{R}$*. For large enough* n *we always obtain* $\lceil na\rceil\leq\lfloor nb\rfloor$*. Also* $a\leq\frac{k}{n}\leq b$*, iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$*.*

In general it holds (by $\sum\limits_{i=-\infty}^{\infty}\Phi\left(x-i\right)=1$, $\forall\ x\in\mathbb{R}$*) that*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)\leq1. \tag{36}$$

Let $(X,\|\cdot\|)$ be a Banach space.

Definition 8 *Let* $f\in C\left(\left[a,b\right],X\right)$ *and* $n\in\mathbb{N}:\lceil na\rceil\leq\lfloor nb\rfloor$*. We introduce and define the* X*-valued linear neural network operators*

$$A_n\left(f,x\right):=\frac{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Phi\left(nx-k\right)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)},\quad x\in\left[a,b\right]. \tag{37}$$

Clearly here $A_n(f,x) \in C([a,b],X)$.

For convenience we use the same A_n for real valued functions when needed. We study here the pointwise and uniform convergence of $A_n(f,x)$ to $f(x)$ with rates.

For convenience, also we call

$$A_n^*(f,x) := \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) \Phi(nx-k), \tag{38}$$

(similarly, A_n^* can be defined for real valued functions) that is

$$A_n(f,x) := \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}. \tag{39}$$

So that

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} - f(x) =$$

$$\frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}. \tag{40}$$

Consequently, we derive that

$$\|A_n(f,x) - f(x)\| \leq 2\left(\sqrt[2m]{1+4^m}\right) \left\| A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)\right) \right\| =$$

$$2\left(\sqrt[2m]{1+4^m}\right) \left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x)\right) \Phi(nx-k) \right\|. \tag{41}$$

We will estimate the right and hand side of (41).

For that we need, for $f \in C([a,b],X)$ the first modulus of continuity

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \|f(x) - f(y)\|, \quad \delta > 0.$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R},X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into X), for $f \in C_B(\mathbb{R},X)$ (continuous and bounded X-valued), and for $f \in C_u(\mathbb{R},X)$ (uniformly continuous).

The fact $f \in C([a,b], X)$ or $f \in C_u(\mathbb{R}, X)$, is equivalent to $\lim_{\delta \to 0} \omega_1(f, \delta) = 0$, see [11].

We make

Definition 9 *When $f \in C_{uB}(\mathbb{R}, X)$, or $f \in C_B(\mathbb{R}, X)$, we define*

$$\overline{A_n}(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k), \tag{42}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}$,

the X-valued quasi-interpolation neural network operator.

We make

Remark 10 *We have that*

$$\left\| f\left(\frac{k}{n}\right) \right\| \le \|f\|_{\infty,\mathbb{R}} < +\infty, \tag{43}$$

and

$$\left\| f\left(\frac{k}{n}\right) \right\| \Phi(nx-k) \le \|f\|_{\infty,\mathbb{R}} \Phi(nx-k) \tag{44}$$

and

$$\sum_{k=-\lambda}^{\lambda} \left\| f\left(\frac{k}{n}\right) \right\| \Phi(nx-k) \le \|f\|_{\infty,\mathbb{R}} \left(\sum_{k=-\lambda}^{\lambda} \Phi(nx-k) \right), \tag{45}$$

and finally

$$\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \Phi(nx-k) \le \|f\|_{\infty,\mathbb{R}}, \tag{46}$$

a convergent in $\mathbb{R}$ series.

So, the series $\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| \Phi(nx-k)$ is absolutely convergent in X, hence it is convergent in X and $\overline{A_n}(f,x) \in X$. We denote by $\|f\|_\infty := \sup_{x \in [a,b]} \|f(x)\|$, for $f \in C([a,b], X)$, similarly it is defined for $f \in C_B(\mathbb{R}, X)$.

3 Main Results

We present a set of X-valued neural network approximations to a function given with rates.

Theorem 11 *Let $f \in C([a,b], X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in [a,b]$, $m \in \mathbb{N}$. Then* *i)*

$$\|A_n(f,x)-f(x)\| \leq \left(\sqrt[2m]{1+4^m}\right)\left[2\omega_1\left(f,\frac{1}{n^\alpha}\right)+\frac{\|f\|_\infty}{m\left(n^{1-\alpha}-2\right)^{2m}}\right]=:\lambda_1, \tag{47}$$

and

ii)

$$\|A_n(f)-f\|_\infty \leq \lambda_1. \tag{48}$$

We get that $\lim_{n\to\infty} A_n(f)=f$*, pointwise and uniformly.*

Proof. We see that

$$\begin{aligned}
&\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)\Phi(nx-k)\right\| \leq \\
&\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|\Phi(nx-k) = \\
&\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|\Phi(nx-k)+ \\
&\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|\Phi(nx-k)\leq \\
&\sum_{\substack{k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}}}^{\lfloor nb\rfloor}\omega_1\left(f,\left|\frac{k}{n}-x\right|\right)\Phi(nx-k)+ \\
&2\|f\|_\infty\sum_{\substack{k=-\infty \\ :|k-nx|>n^{1-\alpha}}}^{\infty}\Phi(nx-k)\leq \\
&\omega_1\left(f,\frac{1}{n^\alpha}\right)\sum_{\substack{k=-\infty \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}}}^{\infty}\Phi(nx-k)+
\end{aligned} \tag{49}$$

$$2\|f\|_\infty \sum_{\begin{cases} k=-\infty \\ : |k-nx| > n^{1-\alpha} \end{cases}}^{\infty} \Phi(nx-k) \underset{\text{(by Theorem 4)}}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{\|f\|_\infty}{2m\left(n^{1-\alpha}-2\right)^{2m}}.$$

That is

$$\left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) \Phi(nx-k) \right\| \leq$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{\|f\|_\infty}{2m\left(n^{1-\alpha}-2\right)^{2m}}. \tag{50}$$

Using (41) we derive (47). ∎

It follows

Theorem 12 *Let $f \in C_B(\mathbb{R}, X)$, $0 < \alpha < 1$, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $x \in \mathbb{R}$, $m \in \mathbb{N}$. Then*

i)

$$\left\|\overline{A_n}(f,x) - f(x)\right\| \leq \omega_1\left(f, \frac{1}{n^\alpha}\right) + \frac{\|f\|_\infty}{2m\left(n^{1-\alpha}-2\right)^{2m}} =: \lambda_2, \tag{51}$$

and

ii)

$$\left\|\overline{A_n}(f) - f\right\|_\infty \leq \lambda_2. \tag{52}$$

For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n\to\infty} \overline{A_n}(f) = f$, pointwise and uniformly.

Proof. We observe that

$$\left\|\overline{A_n}(f,x) - f(x)\right\| = \left\| \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) \Phi(nx-k) - f(x) \sum_{k=-\infty}^{\infty} \Phi(nx-k) \right\| =$$

$$\left\| \sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) \Phi(nx-k) \right\| \leq$$

$$\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \Phi(nx-k) = \tag{53}$$

$$\sum_{\begin{cases} k=-\infty \\ : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha} \end{cases}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \Phi(nx-k) +$$

$$\sum_{\substack{k=-\infty \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| \Phi(nx-k) \leq$$

$$\sum_{\substack{k=-\infty \\ : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} \omega_1\left(f, \left|\frac{k}{n}-x\right|\right) \Phi(nx-k) +$$

$$2\|f\|_{\infty} \sum_{\substack{k=-\infty \\ : \left|\frac{k}{n}-x\right| > \frac{1}{n^{\alpha}}}}^{\infty} \Phi(nx-k) \leq$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) \sum_{\substack{k=-\infty \\ : \left|\frac{k}{n}-x\right| \leq \frac{1}{n^{\alpha}}}}^{\infty} \Phi(nx-k) + \frac{2\|f\|_{\infty}}{4m\left(n^{1-\alpha}-2\right)^{2m}} \leq \tag{54}$$

$$\omega_1\left(f, \frac{1}{n^{\alpha}}\right) + \frac{\|f\|_{\infty}}{2m\left(n^{1-\alpha}-2\right)^{2m}},$$

proving the claim. ■

We need the X-valued Taylor's formula in an appropiate form:

Theorem 13 *([10], [12]) Let $N \in \mathbb{N}$, and $f \in C^N([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space. Let any $x, y \in [a,b]$. Then*

$$f(x) = \sum_{i=0}^{N} \frac{(x-y)^i}{i!} f^{(i)}(y) + \frac{1}{(N-1)!} \int_y^x (x-t)^{N-1} \left(f^{(N)}(t) - f^{(N)}(y)\right) dt. \tag{55}$$

The derivatives $f^{(i)}$, $i \in \mathbb{N}$, are defined like the numerical ones, see [20], p. 83. The integral $\int_y^x$ in (55) is of Bochner type, see [18].

By [12], [16] we have that: if $f \in C([a,b], X)$, then $f \in L_{\infty}([a,b], X)$ and $f \in L_1([a,b], X)$.

In the next we discuss high order neural network X-valued approximation by using the smoothness of f.

Theorem 14 *Let $f \in C^N([a,b], X)$, $n, N, m \in \mathbb{N}$, $0 < \alpha < 1$, $x \in [a,b]$ and $n^{1-\alpha} > 2$. Then*

i)

$$\|A_n(f,x) - f(x)\| \leq \left(\sqrt[2m]{1+4^m}\right) \left\{ \sum_{j=1}^{N} \frac{\left\|f^{(j)}(x)\right\|}{j!} \left[\frac{2}{n^{\alpha j}} + \frac{(b-a)^j}{2m\left(n^{1-\alpha}-2\right)^{2m}}\right] + \right. \tag{56}$$

$$\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{2}{n^{\alpha N}N!}+\frac{\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!m\left(n^{1-\alpha}-2\right)^{2m}}\right]\Bigg\},$$

ii) assume further $f^{(j)}(x_0)=0$, $j=1,...,N$, *for some* $x_0\in[a,b]$, *it holds*

$$\|A_n(f,x_0)-f(x_0)\|\leq\left(\sqrt[2m]{1+4^m}\right)\cdot \tag{57}$$

$$\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{2}{n^{\alpha N}N!}+\frac{\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!m\left(n^{1-\alpha}-2\right)^{2m}}\right],$$

and

iii)

$$\|A_n(f)-f\|_{\infty}\leq\left(\sqrt[2m]{1+4^m}\right)\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left[\frac{2}{n^{\alpha j}}+\frac{(b-a)^j}{2m\left(n^{1-\alpha}-2\right)^{2m}}\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{2}{n^{\alpha N}N!}+\frac{\left\|f^{(N)}\right\|_{\infty}(b-a)^N}{N!m\left(n^{1-\alpha}-2\right)^{2m}}\right]\right\}. \tag{58}$$

We derive that $\lim_{n\to\infty}A_n(f)=f$, *pointwise and uniformly.*

Proof. Next we apply the X-valued Taylor's formula with Bochner integral remainder (55). We have (here $\frac{k}{n},x\in[a,b]$)

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^j+\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt. \tag{59}$$

Then

$$f\left(\frac{k}{n}\right)\Phi(nx-k)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\Phi(nx-k)\left(\frac{k}{n}-x\right)^j+ \tag{60}$$

$$\Phi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Hence

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(\frac{k}{n}\right)\Phi(nx-k)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)= \tag{61}$$

$$\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\left(\frac{k}{n}-x\right)^j+$$

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t)-f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt.$$

Thus

$$A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right) =$$

$$\sum_{j=1}^{N}\frac{f^{(j)}(x)}{j!}A_n^*\left((\cdot - x)^j\right) + \Lambda_n(x), \tag{62}$$

where

$$\Lambda_n(x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\int_x^{\frac{k}{n}}\left(f^{(N)}(t) - f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt. \tag{63}$$

We assume that $b-a > \frac{1}{n^\alpha}$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\alpha}}\right\rceil$.

Thus $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}$ or $\left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}$.

Let

$$\gamma := \int_x^{\frac{k}{n}}\left(f^{(N)}(t) - f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt, \tag{64}$$

in the case of $\left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}$, we find that

$$\|\gamma\| \leq \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!} \tag{65}$$

for $x \leq \frac{k}{n}$ or $x \geq \frac{k}{n}$.

We prove it next.

i) Indeed, for the case of $x \leq \frac{k}{n}$, we have

$$\|\gamma\| = \left\|\int_x^{\frac{k}{n}}\left(f^{(N)}(t) - f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right\| \leq$$

$$\int_x^{\frac{k}{n}}\left\|f^{(N)}(t) - f^{(N)}(x)\right\|\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt \leq$$

$$\int_x^{\frac{k}{n}}\omega_1\left(f^{(N)}, |t-x|\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt \leq \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\int_x^{\frac{k}{n}}\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt = \tag{66}$$

$$\omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{\left(\frac{k}{n}-x\right)^N}{N!} \leq \omega_1\left(f^{(N)}, \frac{1}{n^\alpha}\right)\frac{1}{n^{\alpha N}N!}.$$

ii) for the case of $x > \frac{k}{n}$, we have

$$\|\gamma\| = \left\|\int_x^{\frac{k}{n}}\left(f^{(N)}(t) - f^{(N)}(x)\right)\frac{\left(\frac{k}{n}-t\right)^{N-1}}{(N-1)!}dt\right\| =$$

$$\left\| \int_{\frac{k}{n}}^{x} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \right\| \leq$$

$$\int_{\frac{k}{n}}^{x} \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \leq \tag{67}$$

$$\int_{\frac{k}{n}}^{x} \omega_1 \left(f^{(N)}, |t - x| \right) \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \leq \omega_1 \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \int_{\frac{k}{n}}^{x} \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt =$$

$$\omega_1 \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{\left(x - \frac{k}{n}\right)^{N}}{N!} \leq \omega_1 \left(f^{(N)}, \frac{1}{n^{\alpha}} \right) \frac{1}{n^{\alpha N} N!}.$$

We have proved (65).

We treat again γ, see (64), but differently:

Notice also for $x \leq \frac{k}{n}$ that

$$\left\| \int_{x}^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \right\| \leq$$

$$\int_{x}^{\frac{k}{n}} \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \leq \tag{68}$$

$$2 \left\| f^{(N)} \right\|_{\infty} \int_{x}^{\frac{k}{n}} \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt = 2 \left\| f^{(N)} \right\|_{\infty} \frac{\left(\frac{k}{n} - x\right)^{N}}{N!}$$

$$\leq 2 \left\| f^{(N)} \right\|_{\infty} \frac{(b-a)^{N}}{N!}.$$

Next assume $\frac{k}{n} \leq x$, then

$$\left\| \int_{x}^{\frac{k}{n}} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(\frac{k}{n} - t\right)^{N-1}}{(N-1)!} dt \right\| =$$

$$\left\| \int_{\frac{k}{n}}^{x} \left(f^{(N)}(t) - f^{(N)}(x) \right) \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \right\| \leq$$

$$\int_{\frac{k}{n}}^{x} \left\| f^{(N)}(t) - f^{(N)}(x) \right\| \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt \leq \tag{69}$$

$$2 \left\| f^{(N)} \right\|_{\infty} \int_{\frac{k}{n}}^{x} \frac{\left(t - \frac{k}{n}\right)^{N-1}}{(N-1)!} dt = 2 \left\| f^{(N)} \right\|_{\infty} \frac{\left(x - \frac{k}{n}\right)^{N}}{N!}$$

$$\leq 2 \left\| f^{(N)} \right\|_{\infty} \frac{(b-a)^{N}}{N!}.$$

Thus

$$\|\gamma\| \leq 2\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^{N}}{N!}. \tag{70}$$

in the two cases.

Therefore

$$\Lambda_{n}(x)=\sum_{\substack{\left\{\begin{array}{l}k=\lceil na\rceil\\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{array}\right.}}^{\lfloor nb\rfloor}\Phi(nx-k)\gamma+\sum_{\substack{\left\{\begin{array}{l}k=\lceil na\rceil\\ \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}\end{array}\right.}}^{\lfloor nb\rfloor}\Phi(nx-k)\gamma. \tag{71}$$

Hence

$$\|\Lambda_{n}(x)\| \leq \sum_{\substack{\left\{\begin{array}{l}k=\lceil na\rceil\\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{array}\right.}}^{\lfloor nb\rfloor}\Phi(nx-k)\left(\omega_{1}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{N!n^{\alpha N}}\right)+ \tag{72}$$

$$\left(\sum_{\substack{\left\{\begin{array}{l}k=\lceil na\rceil\\ \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}\end{array}\right.}}^{\lfloor nb\rfloor}\Phi(nx-k)\right)2\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^{N}}{N!}\overset{(19)}{\leq}$$

$$\omega_{1}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{N!n^{\alpha N}}+\frac{1}{4m\left(n^{1-\alpha}-2\right)^{2m}}2\left\|f^{(N)}\right\|_{\infty}\frac{(b-a)^{N}}{N!}=$$

$$\omega_{1}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{N!n^{\alpha N}}+\frac{\left\|f^{(N)}\right\|_{\infty}(b-a)^{N}}{N!2m\left(n^{1-\alpha}-2\right)^{2m}}.$$

That is

$$\|\Lambda_{n}(x)\| \leq \frac{\omega_{1}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)}{N!n^{\alpha N}}+\frac{\left\|f^{(N)}\right\|_{\infty}(b-a)^{N}}{N!2m\left(n^{1-\alpha}-2\right)^{2m}}, \tag{73}$$

$\forall\, x\in[a,b]$.

We further see that

$$A_{n}^{*}\left((\cdot-x)^{j}\right)=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j}, \tag{74}$$

where A_{n}^{*} is defined similarly for real valued functions.

Therefore

$$\left|A_{n}^{*}\left((\cdot-x)^{j}\right)\right| \leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\left|\frac{k}{n}-x\right|^{j}=$$

$$\sum_{\begin{cases} k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb\rfloor} \Phi\left(nx-k\right)\left|\frac{k}{n}-x\right|^{j}+\sum_{\begin{cases} k=\lceil na\rceil \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}} \end{cases}}^{\lfloor nb\rfloor} \Phi\left(nx-k\right)\left|\frac{k}{n}-x\right|^{j}\leq$$

$$\frac{1}{n^{\alpha j}}+\left(b-a\right)^{j}\frac{1}{4m\left(n^{1-\alpha}-2\right)^{2m}}. \tag{75}$$

That is

$$\left|A_{n}^{*}\left(\left(\cdot-x\right)^{j}\right)\right|\leq\frac{1}{n^{\alpha j}}+\left(b-a\right)^{j}\frac{1}{4m\left(n^{1-\alpha}-2\right)^{2m}}, \tag{76}$$

for $j=1,...,N$.

Putting things together we have proved

$$\left\|A_{n}^{*}\left(f,x\right)-f\left(x\right)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi\left(nx-k\right)\right)\right\|\leq\sum_{j=1}^{N}\frac{\left\|f^{(j)}\left(x\right)\right\|}{j!} \tag{77}$$

$$\left[\frac{1}{n^{\alpha j}}+\frac{\left(b-a\right)^{j}}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right]+\left[\omega_{1}\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{\left\|f^{(N)}\right\|_{\infty}\left(b-a\right)^{N}}{N!2m\left(n^{1-\alpha}-2\right)^{2m}}\right],$$

that is establishing the theorem. ■

All integrals from now on are of Bochner type [18].

We need

Definition 15 *([12]) Let $[a,b]\subset\mathbb{R}$, X be a Banach space, $\alpha>0$; $m=\lceil\alpha\rceil\in\mathbb{N}$, ($\lceil\cdot\rceil$ is the ceiling of the number), $f:[a,b]\to X$. We assume that $f^{(m)}\in L_{1}\left([a,b],X\right)$. We call the Caputo-Bochner left fractional derivative of order α:*

$$\left(D_{*a}^{\alpha}f\right)\left(x\right):=\frac{1}{\Gamma\left(m-\alpha\right)}\int_{a}^{x}\left(x-t\right)^{m-\alpha-1}f^{(m)}\left(t\right)dt,\quad\forall\ x\in\left[a,b\right]. \tag{78}$$

If $\alpha\in\mathbb{N}$, we set $D_{*a}^{\alpha}f:=f^{(m)}$ the ordinary X-valued derivative (defined similar to numerical one, see [20], p. 83), and also set $D_{*a}^{0}f:=f$.

By [12], $\left(D_{*a}^{\alpha}f\right)(x)$ exists almost everywhere in $x\in[a,b]$ and $D_{*a}^{\alpha}f\in L_{1}\left([a,b],X\right)$.

If $\left\|f^{(m)}\right\|_{L_{\infty}([a,b],X)}<\infty$, then by [12], $D_{*a}^{\alpha}f\in C\left([a,b],X\right)$, hence $\left\|D_{*a}^{\alpha}f\right\|\in C\left([a,b]\right)$.

We mention

Lemma 16 *([11]) Let $\alpha>0$, $\alpha\notin\mathbb{N}$, $m=\lceil\alpha\rceil$, $f\in C^{m-1}\left([a,b],X\right)$ and $f^{(m)}\in L_{\infty}\left([a,b],X\right)$. Then $D_{*a}^{\alpha}f\left(a\right)=0$.*

We mention

Definition 17 *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\, x \in [a,b]. \tag{79}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [10], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We need

Lemma 18 *([11]) Let $f \in C^{m-1}([a,b], X)$, $f^{(m)} \in L_\infty([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha} f(b) = 0$.*

We mention the left fractional Taylor formula

Theorem 19 *([12]) Let $m \in \mathbb{N}$ and $f \in C^m([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\alpha > 0 : m = \lceil \alpha \rceil$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} \left(D_{*a}^{\alpha} f\right)(z)\, dz, \tag{80}$$

$\forall\, x \in [a,b]$.

We also mention the right fractional Taylor formula

Theorem 20 *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^m([a,b], X)$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(z)\, dz, \tag{81}$$

$\forall\, x \in [a,b]$.

Convention 21 *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{82}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{83}$$

for all $x, x_0 \in [a,b]$.

We mention

Proposition 22 *([11]) Let $f \in C^n([a,b],X)$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

Proposition 23 *([11]) Let $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

We also mention

Proposition 24 *([11]) Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\,dt, \tag{84}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.

Proposition 25 *([11]) Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta-x)^{m-\alpha-1} f^{(m)}(\zeta)\,d\zeta, \tag{85}$$

for all $x, x_0 \in [a,b] : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

Corollary 26 *([11]) Let $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{a} f(x)$, $D_{x_0-}^{a} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into X, X is a Banach space.*

We need

Theorem 27 *([11]) Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Consider*

$$G(x) = \omega_1(f(\cdot,x), \delta, [x,b]), \tag{86}$$

$\delta > 0$, $x \in [a,b]$.

Then G is continuous on $[a,b]$.

Theorem 28 *([11]) Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Then*

$$H(x) = \omega_1(f(\cdot,x), \delta, [a,x]), \tag{87}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

We make

Remark 29 *([11]) Let* $f \in C^{n-1}([a,b])$*,* $f^{(n)} \in L_{\infty}([a,b])$*,* $n = \lceil \nu \rceil$*,* $\nu > 0$*,* $\nu \notin \mathbb{N}$*. Then*

$$\|D_{*a}^{\nu} f(x)\| \leq \frac{\left\|f^{(n)}\right\|_{L_{\infty}([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \quad \forall\, x \in [a,b]. \tag{88}$$

Thus we observe ($\delta > 0$)

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left\|D_{*a}^{\nu} f(x) - D_{*a}^{\nu} f(y)\right\| \leq \tag{89}$$

$$\sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\left\|f^{(n)}\right\|_{L_{\infty}([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu} + \frac{\left\|f^{(n)}\right\|_{L_{\infty}([a,b],X)}}{\Gamma(n-\nu+1)} (y-a)^{n-\nu} \right)$$

$$\leq \frac{2\left\|f^{(n)}\right\|_{L_{\infty}([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}.$$

Consequently

$$\omega_1\left(D_{*a}^{\nu} f, \delta\right) \leq \frac{2\left\|f^{(n)}\right\|_{L_{\infty}([a,b],X)}}{\Gamma(n-\nu+1)} (b-a)^{n-\nu}. \tag{90}$$

Similarly, let $f \in C^{m-1}([a,b])$*,* $f^{(m)} \in L_{\infty}([a,b])$*,* $m = \lceil \alpha \rceil$*,* $\alpha > 0$*,* $\alpha \notin \mathbb{N}$*, then*

$$\omega_1\left(D_{b-}^{\alpha} f, \delta\right) \leq \frac{2\left\|f^{(m)}\right\|_{L_{\infty}([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{91}$$

So for $f \in C^{m-1}([a,b])$*,* $f^{(m)} \in L_{\infty}([a,b])$*,* $m = \lceil \alpha \rceil$*,* $\alpha > 0$*,* $\alpha \notin \mathbb{N}$*, we find*

$$\sup_{x_0 \in [a,b]} \omega_1\left(D_{*x_0}^{\alpha} f, \delta\right)_{[x_0,b]} \leq \frac{2\left\|f^{(m)}\right\|_{L_{\infty}([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}, \tag{92}$$

and

$$\sup_{x_0 \in [a,b]} \omega_1\left(D_{x_0-}^{\alpha} f, \delta\right)_{[a,x_0]} \leq \frac{2\left\|f^{(m)}\right\|_{L_{\infty}([a,b],X)}}{\Gamma(m-\alpha+1)} (b-a)^{m-\alpha}. \tag{93}$$

By [12] we get that $D_{*x_0}^{\alpha} f \in C([x_0,b], X)$, and by [10] we obtain that $D_{x_0-}^{\alpha} f \in C([a,x_0], X)$.

We present the following X-valued fractional approximation result by neural networks.

Theorem 30 *Let $\alpha > 0$, $N = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $f \in C^N([a,b], X)$, $0 < \beta < 1$, $m \in \mathbb{N}$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. Then*

i)

$$\left\| A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j \right)(x) - f(x) \right\| \leq$$

$$\frac{2\left(\sqrt[2m]{1+4^m}\right)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \quad (94)$$

ii) if $f^{(j)}(x) = 0$, for $j = 1, ..., N-1$, we have

$$\| A_n(f,x) - f(x) \| \leq \frac{2\left(\sqrt[2m]{1+4^m}\right)}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\}, \quad (95)$$

iii)

$$\| A_n(f,x) - f(x) \| \leq 2\left(\sqrt[2m]{1+4^m}\right) \cdot$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)}(x) \right\|}{j!} \left\{ \frac{1}{n^{\beta j}} + \frac{(b-a)^j}{4m\left(n^{1-\beta}-2\right)^{2m}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^\alpha f, \frac{1}{n^\beta}\right)_{[a,x]} + \omega_1\left(D_{*x}^\alpha f, \frac{1}{n^\beta}\right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left.\left. \frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}} \left(\left\| D_{x-}^\alpha f \right\|_{\infty,[a,x]} (x-a)^\alpha + \left\| D_{*x}^\alpha f \right\|_{\infty,[x,b]} (b-x)^\alpha \right) \right\} \right\}, \quad (96)$$

$\forall\, x \in [a,b]$,

and

iv)

$$\| A_n f - f \|_\infty \leq 2\left(\sqrt[2m]{1+4^m}\right) \cdot$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_\infty}{j!} \left\{ \frac{1}{n^{\beta j}} + \frac{(b-a)^j}{4m\left(n^{1-\beta}-2\right)^{2m}} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\frac{(b-a)^{\alpha}}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\sup_{x\in[a,b]}\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}\right)\right\}\Bigg\}. \quad (97)$$

Above, when $N=1$ the sum $\sum_{j=1}^{N-1}\cdot=0$.

As we see here we obtain X-valued fractionally type pointwise and uniform convergence with rates of $A_n\to I$ the unit operator, as $n\to\infty$.

Proof. Let $x\in[a,b]$. We have that $D^{\alpha}_{x-}f(x)=D^{\alpha}_{*x}f(x)=0$.

From Theorem 19, we get by the left Caputo fractional Taylor formula that

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \quad (98)$$

$$\frac{1}{\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D^{\alpha}_{*x}f(J)-D^{\alpha}_{*x}f(x)\right)dJ,$$

for all $x\le\frac{k}{n}\le b$.

Also from Theorem 20, using the right Caputo fractional Taylor formula we get

$$f\left(\frac{k}{n}\right)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\left(\frac{k}{n}-x\right)^{j}+ \quad (99)$$

$$\frac{1}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D^{\alpha}_{x-}f(J)-D^{\alpha}_{x-}f(x)\right)dJ,$$

for all $a\le\frac{k}{n}\le x$.

Hence we have

$$f\left(\frac{k}{n}\right)\Phi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \quad (100)$$

$$\frac{\Phi(nx-k)}{\Gamma(\alpha)}\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D^{\alpha}_{*x}f(J)-D^{\alpha}_{*x}f(x)\right)dJ,$$

for all $x\le\frac{k}{n}\le b$, iff $\lceil nx\rceil\le k\le\lfloor nb\rfloor$, and

$$f\left(\frac{k}{n}\right)\Phi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \quad (101)$$

$$\frac{\Phi(nx-k)}{\Gamma(\alpha)}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ,$$

for all $a \leq \frac{k}{n} \leq x$, iff $\lceil na \rceil \leq k \leq \lfloor nx \rfloor$.

Therefore it holds

$$\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \tag{102}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor}\Phi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ,$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k)=\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j}+ \tag{103}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor}\Phi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ.$$

Adding the last two equalities (102) and (103) obtain

$$A_n^*(f,x)=\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right)\Phi(nx-k)= \tag{104}$$

$$\sum_{j=0}^{N-1}\frac{f^{(j)}(x)}{j!}\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\Phi(nx-k)\left(\frac{k}{n}-x\right)^{j}+$$

$$\frac{1}{\Gamma(\alpha)}\left\{\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor}\Phi(nx-k)\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\left(D_{x-}^{\alpha}f(J)-D_{x-}^{\alpha}f(x)\right)dJ+\right.$$

$$\left.\sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor}\Phi(nx-k)\int_{x}^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right\}.$$

So we have derived

$$A_n^*(f,x)-f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor}\Phi(nx-k)\right)= \tag{105}$$

$$\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}A_n^*\left((\cdot-x)^{j}\right)+u_n(x),$$

where

$$u_n(x) := \frac{1}{\Gamma(\alpha)} \left\{ \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x) \right) dJ \right.$$

$$\left. + \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k) \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x) \right) dJ \right\}. \quad (106)$$

We set

$$u_{1n}(x) := \frac{1}{\Gamma(\alpha)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x) \right) dJ, \quad (107)$$

and

$$u_{2n} := \frac{1}{\Gamma(\alpha)} \sum_{k=\lceil nx \rceil+1}^{\lfloor nb \rfloor} \Phi(nx-k) \int_{x}^{\frac{k}{n}} \left(\frac{k}{n} - J \right)^{\alpha-1} \left(D_{*x}^{\alpha} f(J) - D_{*x}^{\alpha} f(x) \right) dJ, \quad (108)$$

i.e.

$$u_n(x) = u_{1n}(x) + u_{2n}(x). \quad (109)$$

We assume $b-a > \frac{1}{n^{\beta}}$, $0<\beta<1$, which is always the case for large enough $n \in \mathbb{N}$, that is when $n > \left\lceil (b-a)^{-\frac{1}{\beta}} \right\rceil$. It is always true that either $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ or $\left| \frac{k}{n} - x \right| > \frac{1}{n^{\beta}}$.

For $k = \lceil na \rceil, ..., \lfloor nx \rfloor$, we consider

$$\gamma_{1k} := \left\| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left(D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x) \right) dJ \right\| = \quad (110)$$

$$\left\| \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} D_{x-}^{\alpha} f(J)\, dJ \right\| \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left\| D_{x-}^{\alpha} f(J) \right\| dJ \leq$$

$$\left\| D_{x-}^{\alpha} f(J) \right\|_{\infty,[a,x]} \frac{\left(x - \frac{k}{n} \right)^{\alpha}}{\alpha} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}. \quad (111)$$

That is

$$\gamma_{1k} \leq \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} \frac{(x-a)^{\alpha}}{\alpha}, \quad (112)$$

for $k = \lceil na \rceil, ..., \lfloor nx \rfloor$.

Also we have in case of $\left| \frac{k}{n} - x \right| \leq \frac{1}{n^{\beta}}$ that

$$\gamma_{1k} \leq \int_{\frac{k}{n}}^{x} \left(J - \frac{k}{n} \right)^{\alpha-1} \left\| D_{x-}^{\alpha} f(J) - D_{x-}^{\alpha} f(x) \right\| dJ \leq$$

$$\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}\omega_1\left(D_{x-}^{\alpha}f,|J-x|\right)_{[a,x]}dJ\leq \tag{113}$$

$$\omega_1\left(D_{x-}^{\alpha}f,\left|x-\frac{k}{n}\right|\right)_{[a,x]}\int_{\frac{k}{n}}^{x}\left(J-\frac{k}{n}\right)^{\alpha-1}dJ\leq$$

$$\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\frac{\left(x-\frac{k}{n}\right)^{\alpha}}{\alpha}\leq\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\frac{1}{\alpha n^{\alpha\beta}}.$$

That is when $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}$, then

$$\gamma_{1k}\leq\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{\alpha n^{\alpha\beta}}. \tag{114}$$

Consequently we obtain

$$\|u_{1n}(x)\|\leq\frac{1}{\Gamma(\alpha)}\sum_{k=\lceil na\rceil}^{\lfloor nx\rfloor}\Phi(nx-k)\,\gamma_{1k}= \tag{115}$$

$$\frac{1}{\Gamma(\alpha)}\left\{\sum_{\begin{cases}k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Phi(nx-k)\,\gamma_{1k}+\sum_{\begin{cases}k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Phi(nx-k)\,\gamma_{1k}\right\}\leq$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left(\sum_{\begin{cases}k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Phi(nx-k)\right)\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{\alpha n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\begin{cases}k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\end{cases}}^{\lfloor nx\rfloor}\Phi(nx-k)\right)\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\frac{(x-a)^{\alpha}}{\alpha}\right\}\leq \tag{116}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\begin{cases}k=-\infty\\ :|nx-k|>n^{1-\beta}\end{cases}}^{\infty}\Phi(nx-k)\right)\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}\right\}\leq$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}}{4m\left(n^{1-\beta}-2\right)^{2m}}\right\}.$$

So we have proved that

$$\|u_{1n}(x)\|\leq\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}}{n^{\alpha\beta}}+\frac{\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}}{4m\left(n^{1-\beta}-2\right)^{2m}}\right\}. \tag{117}$$

Next when $k=\lfloor nx\rfloor+1,...,\lfloor nb\rfloor$ we consider

$$\gamma_{2k}:=\left\|\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left(D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right)dJ\right\|\leq \tag{118}$$

$$\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left\|D_{*x}^{\alpha}f(J)-D_{*x}^{\alpha}f(x)\right\|dJ=$$

$$\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\left\|D_{*x}^{\alpha}f(J)\right\|dJ\leq$$

$$\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{\left(\frac{k}{n}-x\right)^{\alpha}}{\alpha}\leq\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}. \tag{119}$$

Therefore when $k=\lfloor nx\rfloor+1,...,\lfloor nb\rfloor$ we get that

That is

$$\gamma_{2k}\leq\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}. \tag{120}$$

In case of $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}$ we have

$$\gamma_{2k}\leq\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}\omega_1\left(D_{*x}^{\alpha}f,|J-x|\right)_{[x,b]}dJ\leq$$

$$\omega_1\left(D_{*x}^{\alpha}f,\left|\frac{k}{n}-x\right|\right)_{[x,b]}\int_x^{\frac{k}{n}}\left(\frac{k}{n}-J\right)^{\alpha-1}dJ\leq \tag{121}$$

$$\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\frac{\left(\frac{k}{n}-x\right)^{\alpha}}{\alpha}\leq\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\frac{1}{\alpha n^{\alpha\beta}}.$$

So when $\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}$ we derived that

$$\gamma_{2k}\leq\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{\alpha n^{\alpha\beta}}. \tag{122}$$

Similarly we have that

$$\|u_{2n}(x)\|\leq\frac{1}{\Gamma(\alpha)}\left(\sum_{k=\lfloor nx\rfloor+1}^{\lfloor nb\rfloor}\Phi(nx-k)\gamma_{2k}\right)=$$

$$\frac{1}{\Gamma(\alpha)}\left\{\sum_{\substack{\left\{ k=\lfloor nx\rfloor+1 \\ : \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\right.}}^{\lfloor nb\rfloor}\Phi(nx-k)\,\gamma_{2k}+\sum_{\substack{\left\{ k=\lfloor nx\rfloor+1 \\ : \left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\right.}}^{\lfloor nb\rfloor}\Phi(nx-k)\,\gamma_{2k}\right\}\leq \tag{123}$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left(\sum_{\substack{\left\{ k=\lfloor nx\rfloor+1 \\ : \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\beta}}\right.}}^{\lfloor nb\rfloor}\Phi(nx-k)\right)\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{\alpha n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\substack{\left\{ k=\lfloor nx\rfloor+1 \\ : \left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\right.}}^{\lfloor nb\rfloor}\Phi(nx-k)\right)\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\frac{(b-x)^{\alpha}}{\alpha}\right\}\leq$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right.$$

$$\left.\left(\sum_{\substack{\left\{ k=-\infty \\ : \left|\frac{k}{n}-x\right|>\frac{1}{n^{\beta}}\right.}}^{\infty}\Phi(nx-k)\right)\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right\}\leq \tag{124}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}}{4m\left(n^{1-\beta}-2\right)^{2m}}\right\}.$$

So we have proved that

$$\left\|u_{2n}(x)\right\|\leq\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\frac{\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}}{4m\left(n^{1-\beta}-2\right)^{2m}}\right\}. \tag{125}$$

Therefore

$$\left\|u_n(x)\right\|\leq\left\|u_{1n}(x)\right\|+\left\|u_{2n}(x)\right\|\leq$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}}+\right. \tag{126}$$

$$\left.\frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}.$$

From the proof of Theorem 14 we get that

$$\left|A_n^*\left((\cdot - x)^j\right)(x)\right| \leq \frac{1}{n^{\beta j}} + \frac{(b-a)^j}{4m\left(n^{1-\beta}-2\right)^{2m}}, \tag{127}$$

for $j = 1, ..., N-1$, $\forall\ x \in [a,b]$.

Putting things together, we have established

$$\left\|A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right)\right\| \leq \sum_{j=1}^{N-1}\frac{\left\|f^{(j)}(x)\right\|}{j!} \tag{128}$$

$$\left[\frac{1}{n^{\beta j}} + \frac{(b-a)^j}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right] +$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}}{n^{\alpha\beta}} +\right.$$

$$\left.\frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\} =: K_n(x). \tag{129}$$

As a result we derive (see (41))

$$\left\|A_n(f,x) - f(x)\right\| \leq 2\left(\sqrt[2m]{1+4^m}\right)K_n(x), \quad \forall\ x \in [a,b]. \tag{130}$$

We further have that

$$\left\|K_n\right\|_{\infty} \leq \sum_{j=1}^{N-1}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left[\frac{1}{n^{\beta j}} + (b-a)^j\frac{1}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right] + \tag{131}$$

$$\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left\{\sup_{x\in[a,b]}\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]}\right) + \sup_{x\in[a,b]}\left(\omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)\right\}}{n^{\alpha\beta}} +\right.$$

$$(b-a)^{\alpha}\frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}$$

$$\left.\left\{\left(\sup_{x\in[a,b]}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\right) + \sup_{x\in[a,b]}\left(\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right)\right\}\right\} =: E_n.$$

Hence it holds

$$\left\|A_n f - f\right\|_{\infty} \leq 2\left(\sqrt[2m]{1+4^m}\right)E_n. \tag{132}$$

We observe the following:

We have

$$\left(D_{x-}^{\alpha}f\right)(y)=\frac{(-1)^{N}}{\Gamma(N-\alpha)}\int_{y}^{x}(J-y)^{N-\alpha-1}f^{(N)}(J)\,dJ,\quad\forall\ y\in[a,x]\tag{133}$$

and

$$\begin{aligned}\left\|\left(D_{x-}^{\alpha}f\right)(y)\right\|\leq\frac{1}{\Gamma(N-\alpha)}\left(\int_{y}^{x}(J-y)^{N-\alpha-1}\,dJ\right)\left\|f^{(N)}\right\|_{\infty}=\\ \frac{1}{\Gamma(N-\alpha)}\frac{(x-y)^{N-\alpha}}{(N-\alpha)}\left\|f^{(N)}\right\|_{\infty}=\frac{(x-y)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}\\ \leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}.\end{aligned}\tag{134}$$

That is

$$\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty},\tag{135}$$

and

$$\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}.\tag{136}$$

Similarly we have

$$\left(D_{*x}^{\alpha}f\right)(y)=\frac{1}{\Gamma(N-\alpha)}\int_{x}^{y}(y-t)^{N-\alpha-1}f^{(N)}(t)\,dt,\quad\forall\ y\in[x,b].\tag{137}$$

Thus we get

$$\begin{aligned}\left\|\left(D_{*x}^{\alpha}f\right)(y)\right\|\leq\frac{1}{\Gamma(N-\alpha)}\left(\int_{x}^{y}(y-t)^{N-\alpha-1}\,dt\right)\left\|f^{(N)}\right\|_{\infty}\leq\\ \frac{1}{\Gamma(N-\alpha)}\frac{(y-x)^{N-\alpha}}{(N-\alpha)}\left\|f^{(N)}\right\|_{\infty}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}.\end{aligned}\tag{138}$$

Hence

$$\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty},\tag{139}$$

and

$$\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\leq\frac{(b-a)^{N-\alpha}}{\Gamma(N-\alpha+1)}\left\|f^{(N)}\right\|_{\infty}.\tag{140}$$

From (92) and (93) we get

$$\sup_{x\in[a,b]}\omega_{1}\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}\leq\frac{2\left\|f^{(N)}\right\|_{\infty}}{\Gamma(N-\alpha+1)}(b-a)^{N-\alpha},\tag{141}$$

and

$$\sup_{x\in[a,b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \leq \frac{2 \left\| f^{(N)} \right\|_{\infty}}{\Gamma(N-\alpha+1)} (b-a)^{N-\alpha}. \tag{142}$$

That is $E_n < \infty$.

We finally notice that

$$A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j \right)(x) - f(x) =$$

$$\frac{A_n^*(f,x)}{\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \right)} - \frac{1}{\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \right)} \cdot$$

$$\left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^*\left((\cdot - x)^j \right)(x) \right) - f(x) =$$

$$\frac{1}{\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \right)} \left[A_n^*(f,x) - \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^*\left((\cdot - x)^j \right)(x) \right) \right. \tag{143}$$

$$\left. - \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \right) f(x) \right].$$

Therefore we get

$$\left\| A_n(f,x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n\left((\cdot - x)^j \right)(x) - f(x) \right\| \leq 2 \left(\sqrt[2m]{1+4^m} \right) \cdot$$

$$\left\| A_n^*(f,x) - \left(\sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} A_n^*\left((\cdot - x)^j \right)(x) \right) - \left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \right) f(x) \right\|, \tag{144}$$

$\forall\, x \in [a,b]$.

The proof of the theorem is now completed. ∎

Next we apply Theorem 30 for $N = 1$.

Theorem 31 *Let* $0 < \alpha, \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $m \in \mathbb{N}$. *Then*

i)

$$\| A_n(f,x) - f(x) \| \leq$$

$$\frac{2 \left(\sqrt[2m]{1+4^m} \right)}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1\left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\Bigg\}, \tag{145}$$

and

ii)

$$\|A_n f-f\|_{\infty}\leq\frac{2\left(\sqrt[2m]{1+4^m}\right)}{\Gamma(\alpha+1)}\cdot$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\frac{(b-a)^{\alpha}}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}. \tag{146}$$

When $\alpha=\frac{1}{2}$ we derive

Corollary 32 *Let $0<\beta<1$, $f\in C^1([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$, $m\in\mathbb{N}$. Then*

i)

$$\|A_n(f,x)-f(x)\|\leq$$

$$\frac{4\left(\sqrt[2m]{1+4^m}\right)}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.\frac{1}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}\sqrt{(x-a)}+\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\sqrt{(b-x)}\right)\right\}, \tag{147}$$

and

ii)

$$\|A_n f-f\|_{\infty}\leq\frac{4\left(\sqrt[2m]{1+4^m}\right)}{\sqrt{\pi}}\cdot$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}}+\right.$$

$$\left.\frac{\sqrt{(b-a)}}{4m\left(n^{1-\beta}-2\right)^{2m}}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\right)\right\}<\infty. \tag{148}$$

Next we make

Remark 33 *Some convergence analysis follows:*

Let $0 < \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $m \in \mathbb{N}$. *We elaborate on (148). Assume that*

$$\omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} \leq \frac{K_1}{n^\beta}, \tag{149}$$

and

$$\omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]} \leq \frac{K_2}{n^\beta}, \tag{150}$$

$\forall\, x \in [a,b]$, $\forall\, n \in \mathbb{N}$, *where* $K_1, K_2 > 0$.

Then it holds

$$\frac{\left[\sup\limits_{x\in[a,b]} \omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[a,x]} + \sup\limits_{x\in[a,b]} \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{1}{n^\beta}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}} \leq$$

$$\frac{\frac{(K_1+K_2)}{n^\beta}}{n^{\frac{\beta}{2}}} = \frac{(K_1+K_2)}{n^{\frac{3\beta}{2}}} = \frac{K}{n^{\frac{3\beta}{2}}}, \tag{151}$$

where $K := K_1 + K_2 > 0$.

The other summand of the right hand side of (148), for large enough n, *converges to zero at the speed* $\frac{1}{n^{2m(1-\beta)}}$, *so it is about* $\frac{L}{n^{2m(1-\beta)}}$, *where* $L > 0$ *is a constant.*

Then, for large enough $n \in \mathbb{N}$, *by (148), (151) and the above comment, we obtain that*

$$\|A_n f - f\|_\infty \leq \frac{M}{\min\left(n^{\frac{3\beta}{2}}, n^{2m(1-\beta)}\right)}, \tag{152}$$

where $M > 0$.

Clearly we have two cases:

i)

$$\|A_n f - f\|_\infty \leq \frac{M}{n^{2m(1-\beta)}}, \text{ when } \frac{4m}{3+4m} \leq \beta < 1, \tag{153}$$

with speed of convergence $\frac{1}{n^{2m(1-\beta)}}$,

and

ii)

$$\|A_n f - f\|_\infty \leq \frac{M}{n^{\frac{3\beta}{2}}}, \text{ when } 0 < \beta \leq \frac{4m}{3+4m}, \tag{154}$$

with speed of convergence $\frac{1}{n^{\frac{3\beta}{2}}}$.

In Theorem 11, for $f \in C([a,b], X)$ *and for large enough* $n \in \mathbb{N}$, *when* $0 < \beta \leq \frac{2m}{1+2m}$, *the speed is* $\frac{1}{n^\beta}$. *So when* $0 < \beta \leq \frac{4m}{3+4m}$ $\left(< \frac{2m}{1+2m}\right)$, *we get by*

(154) that $\|A_n f - f\|_\infty$ converges much faster to zero. The last comes because we assumed differentiability of f.

Notice that in Corollary 32 no initial condition is assumed.

Next, we will present an alternative fractional approximation by A_n, $n \in \mathbb{N}$.

Notation 34 *Let $\overline{n} \in \mathbb{N}$, we denote the left iterated fractional derivative*

$$D_{*x}^{\overline{n}x} = D_{*x}^{\alpha} D_{*x}^{\alpha} \dots D_{*x}^{\alpha}, \quad (\overline{n} \text{ - times}), \tag{155}$$

$x \in [a,b]$, $0 < \alpha < 1$.

Similarly, we also denote the right iterated fractional derivative

$$D_{x-}^{\overline{n}x} = D_{x-}^{\alpha} D_{x-}^{\alpha} \dots D_{x-}^{\alpha}, \quad (\overline{n} \text{ - times}), \tag{156}$$

$x \in [a,b]$.

We need

Definition 35 *Let $\overline{n} \in \mathbb{N}$, $D_x^{(\overline{n}+1)\alpha} f$ denote any of $D_{*x}^{(\overline{n}+1)\alpha}$, $D_{x-}^{(\overline{n}+1)\alpha}$, and $\delta > 0$. We set*

$$\omega_1 \left(D_x^{(\overline{n}+1)\alpha} f, \delta \right) = \max \left\{ \omega_1 \left(D_{*x}^{(\overline{n}+1)\alpha} f, \delta \right)_{[x,b]}, \omega_1 \left(D_{x-}^{(\overline{n}+1)\alpha} f, \delta \right)_{[a,x]} \right\}, \tag{157}$$

where $x \in [a,b]$. Here the moduli of continuity are considered over $[x,b]$ and $[a,x]$, respectively.

We also need

Theorem 36 *([13], p. 123) Let $0 < \alpha < 1$, $f : [a,b] \to \mathbb{R}$, $f' \in L_\infty([a,b])$, $x \in [a,b]$ being fixed. Assume that $D_{*x}^{k\alpha} f \in C([x,b])$, $k = 0, 1, \dots, \overline{n}+1$, $\overline{n} \in \mathbb{N}$, and $\left(D_{*x}^{i\alpha} f \right)(x) = 0$, $i = 2, 3, \dots, \overline{n}+1$. Also, suppose that $D_{x-}^{k\alpha} f \in C([a,x])$, for $k = 0, 1, \dots, \overline{n}+1$, and $\left(D_{x-}^{i\alpha} f \right)(x) = 0$, for $i = 2, 3, \dots, \overline{n}+1$. Then*

$$|f(\cdot) - f(x)| \leq \frac{\omega_1 \left(D_x^{(\overline{n}+1)\alpha} f, \delta \right)}{\Gamma((\overline{n}+1)\alpha + 1)} \left[|\cdot - x|^{(\overline{n}+1)\alpha} + \frac{|\cdot - x|^{(\overline{n}+1)\alpha + 1}}{\delta((\overline{n}+1)\alpha + 1)} \right], \quad \delta > 0. \tag{158}$$

We present

Theorem 37 *Let $f \in C([a,b])$ and all as in Theorem 36, $n \in \mathbb{N} : n^{1-\alpha} > 2$, $m \in \mathbb{N}$.*

Then

$$|(A_n f)(x) - f(x)| \leq \frac{2 \left(\sqrt[2m]{1 + 4^m} \right) \omega_1 \left(D_x^{(\overline{n}+1)\alpha} f, \delta \right)}{\Gamma((\overline{n}+1)\alpha + 1)}$$

$$\left\{\left[\frac{1}{n^{(\overline{n}+1)\alpha^2}}+\frac{(b-a)^{(\overline{n}+1)\alpha}}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right]+\right. \tag{159}$$

$$\left.\frac{1}{\delta\left((\overline{n}+1)\alpha+1\right)}\left[\frac{1}{n^{\alpha[(\overline{n}+1)\alpha+1]}}+\frac{(b-a)^{(\overline{n}+1)\alpha+1}}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right]\right\}, \quad \delta>0.$$

Hence $\lim_{n\to+\infty} A_n(f)(x) = f(x)$.

Proof. We notice that A_n is a positive linear operator with $A_n(1)=1$.
Let $f\in C([a,b],\mathbb{R})$, then $|f|\leq|f|$ and $-|f|\leq f\leq|f|$.
Hence $-A_n(|f|)\leq A_n(f)\leq A_n(|f|)$ and $|A_n(f)|\leq A_n(|f|)$.
Therefore

$$|(A_nf)(x)-f(x)| = |(A_nf)(x)-A_n(f(x))(x)| =$$

$$|A_n(f-f(x))(x)| \overset{(158)}{\leq} A_n(|f-f(x)|)(x) \leq \tag{160}$$

$$\frac{\omega_1\left(D_x^{(\overline{n}+1)\alpha}f,\delta\right)}{\Gamma\left((\overline{n}+1)\alpha+1\right)}\left[A_n\left(|\cdot-x|^{(\overline{n}+1)\alpha}\right)(x)+\frac{A_n\left(|\cdot-x|^{(\overline{n}+1)\alpha+1}\right)(x)}{\delta\left((\overline{n}+1)\alpha+1\right)}\right]=$$

$$\frac{\omega_1\left(D_x^{(\overline{n}+1)\alpha}f,\delta\right)}{\Gamma\left((\overline{n}+1)\alpha+1\right)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)}$$

$$\left[\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|\frac{k}{n}-x\right|^{(\overline{n}+1)\alpha}\Phi(nx-k)+\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left|\frac{k}{n}-x\right|^{(\overline{n}+1)\alpha+1}\Phi(nx-k)}{\delta\left((\overline{n}+1)\alpha+1\right)}\right]\overset{(26)}{\leq} \tag{161}$$

$$\frac{2\left(\sqrt[2m]{1+4^m}\right)\omega_1\left(D_x^{(\overline{n}+1)\alpha}f,\delta\right)}{\Gamma\left((\overline{n}+1)\alpha+1\right)}$$

$$\left\{\left[\sum_{\begin{cases}k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}\end{cases}}^{\lfloor nb\rfloor}\left|\frac{k}{n}-x\right|^{(\overline{n}+1)\alpha}\Phi(nx-k)+\right.\right.$$

$$\left.\sum_{\begin{cases}k=\lceil na\rceil\\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^\alpha}\end{cases}}^{\lfloor nb\rfloor}\left|\frac{k}{n}-x\right|^{(\overline{n}+1)\alpha}\Phi(nx-k)\right]+$$

$$\frac{1}{\delta\left(\left(\overline{n}+1\right)\alpha+1\right)}\left[\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}\end{cases}}^{\lfloor nb\rfloor}\left|\frac{k}{n}-x\right|^{\left(\overline{n}+1\right)\alpha+1}\Phi\left(nx-k\right)+\right. \tag{162}$$

$$\left.\left.\sum_{\begin{cases} k=\lceil na\rceil \\ :\left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}\end{cases}}^{\lfloor nb\rfloor}\left|\frac{k}{n}-x\right|^{\left(\overline{n}+1\right)\alpha+1}\Phi\left(nx-k\right)\right]\right\}\overset{(19)}{\leq}$$

$$\frac{2\left(\sqrt[2m]{1+4^{m}}\right)\omega_{1}\left(D_{x}^{\left(\overline{n}+1\right)\alpha}f,\delta\right)}{\Gamma\left(\left(\overline{n}+1\right)\alpha+1\right)}\left\{\left[\frac{1}{n^{\left(\overline{n}+1\right)\alpha^{2}}}+\frac{\left(b-a\right)^{\left(\overline{n}+1\right)\alpha}}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right]\right.$$

$$\left.+\frac{1}{\delta\left(\left(\overline{n}+1\right)\alpha+1\right)}\left[\frac{1}{n^{\alpha\left[\left(\overline{n}+1\right)\alpha+1\right]}}+\frac{\left(b-a\right)^{\left(\overline{n}+1\right)\alpha+1}}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right]\right\},\ \delta>0, \tag{163}$$

proving the claim. ■

We finish with

Corollary 38 *All as in Theorem 37, with* $\delta=\frac{1}{\left(\overline{n}+1\right)\alpha+1}$. *Then*

$$\left|\left(A_{n}f\right)\left(x\right)-f\left(x\right)\right|\leq\frac{2\left(\sqrt[2m]{1+4^{m}}\right)\omega_{1}\left(D_{x}^{\left(\overline{n}+1\right)\alpha}f,\frac{1}{\left(\overline{n}+1\right)\alpha+1}\right)}{\Gamma\left(\left(\overline{n}+1\right)\alpha+1\right)}$$

$$\left\{\left[\frac{1}{n^{\left(\overline{n}+1\right)\alpha^{2}}}+\frac{\left(b-a\right)^{\left(\overline{n}+1\right)\alpha}}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right]+\left[\frac{1}{n^{\alpha\left[\left(\overline{n}+1\right)\alpha+1\right]}}+\frac{\left(b-a\right)^{\left(\overline{n}+1\right)\alpha+1}}{4m\left(n^{1-\alpha}-2\right)^{2m}}\right]\right\}. \tag{164}$$

Hence $\lim_{n\to+\infty}A_{n}\left(f\right)\left(x\right)=f\left(x\right).$

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