Time-dependent conformal mapping techniques applied to fluid sloshing problems

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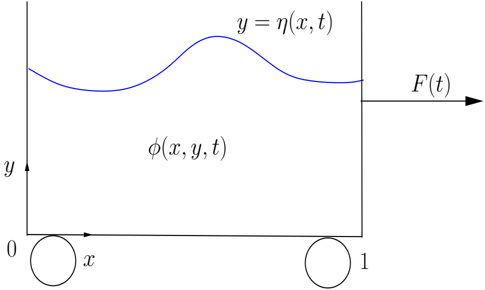
Outline

We are interested in studying the following fluid dynamic problem



Here fluid is inviscid, irrotational and motion is confined to two-dimensions.

$\eta(x,t)$ is the unknown free-surface, to be found as part of the analysis.

$F(t)$ is either an imposed external forcing (given), or a coupled forcing (calculated via additional equation).

In anticipation of using conformal mappings we extend the domain from $x \in [0,1]$ to $x \in [0,2]$ by forming the even extension about $x = 1$.

This gives a periodic domain with period 2 in our notation.

Mass conservation gives

$$\Delta\phi := \phi_{xx} + \phi_{yy} = 0\,, \quad 0 < y < \eta(x,t)\,, \quad 0 < x < 1\,.$$

Kinematic (free-surface particles remain on the free-surface) free-surface condition is

$$\eta_t + \phi_x \eta_x = \phi_y \quad \text{at} \quad y = \eta(x,t) \;\; 0 \leq x \leq 1\,.$$

While the dynamic (continuity of pressure) free-surface condition is:

$$
\begin{aligned}
\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + g(\eta - h_0) + x\ddot{F} &= 0\,, \\
\text{on} \quad y &= \eta(x,t) \quad 0 \le x \le 1 \\
\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_y^2) + g(\eta - h_0) + (2-x)\ddot{F} &= 0\,, \\
\text{on} \quad y &= \eta(x,t) \quad 1 \le x \le 2
\end{aligned}
$$

The boundary conditions at the vessel walls are

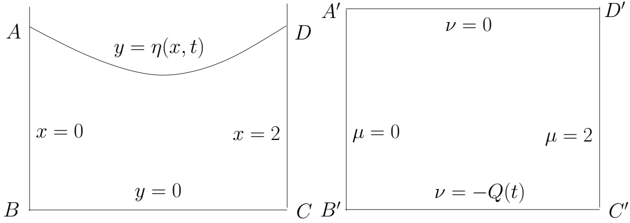
$$
\phi_y = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad \phi_x = 0 \quad \text{at} \quad x = 0, 1, 2\,.
$$

Additional equation for $F(t)$ if required.

To generate fully nonlinear numerical solutions of these equations is difficult due to the unknown position of the free-surface $\eta(x,t)$.

Numerical schemes are easier to implement on fixed domains.

Thus we wish to introduce a mapping such that:



where $Q(t)$ is the conformal modulus which is not known a priori.

Therefore the physical Cartesian coordinates transform to

$$(\mu, \nu, t) \mapsto (x(\mu, \nu, t), y(\mu, \nu, t)) .$$

As the mapping is conformal

$$x_\mu = y_\nu \quad \text{and} \quad x_\nu = -y_\mu ,$$

then both x and y satisfy Laplace's equation

$$x_{\mu\mu} + x_{\nu\nu} = 0, \qquad y_{\mu\mu} + y_{\nu\nu} = 0.$$

On the free-surface

$$(X(\mu, t), Y(\mu, t)) = (x(\mu, 0, t), y(\mu, 0, t)) \quad \text{for} \quad \mu = [0, 2].$$

gives parametric representation of free-surface.

Writing the free-surface equations in parametric form in the rectangular computational domain are

$$X_\mu Y_t - Y_\mu X_t = -\Psi_\mu \,, \quad \text{at} \quad \nu = 0 \,.$$

and

$$\begin{aligned} J\Phi_t - (Y_\mu Y_t + X_\mu X_t)\Phi_\mu + \frac{1}{2}\left(\Phi_\mu^2 - \Psi_\mu^2\right) + gJ(Y - h_0) + X\ddot{F} &= 0\,, \\ \text{at} \quad \nu &= 0\,, \end{aligned}$$

where $J = X_\mu^2 + Y_\mu^2$.

The bottom boundary condition reduces to

$$\psi_\mu = 0 \quad \text{on} \quad \nu = -Q(t).$$

Ultimately, we hope to time-integrate these equations to find X, Y and Φ, but these two equations are implicit for the time derivative terms, hence are difficult to time integrate using Runge-Kutta, say.

But we note that for the parameterised curve $(X(\mu,t), Y(\mu,t))$, any curve can be written as the linear combination of its tangent vector (X_μ, Y_μ) and its normal vector $(-Y_\mu, X_\mu)$, so

$$\begin{aligned} X_t &= -\frac{\beta}{J} Y_\mu + \frac{\alpha}{J} X_\mu \\ Y_t &= \frac{\beta}{J} X_\mu + \frac{\alpha}{J} Y_\mu\,, \end{aligned}$$

where $\beta = -\Psi_\mu$ from the kinematic condition. Here α, representing the tangential fluid velocity, is unknown at this stage but is fixed by forcing the transformation to be analytic.
Also the dynamic condition becomes

$$\Phi_t = -g(Y-\delta) - \frac{1}{2J}\Phi_\mu^2 + \frac{1}{2J}\Psi_\mu^2 + \frac{\alpha}{J}\Phi_\mu.$$

Note: There is no PDE for Ψ hence we cannot directly time integrate these equations, we also have the problem of determining the form of α. However we do have the two complex functions

$$z = x + \mathrm{i}y \qquad w = \phi + \mathrm{i}\psi.$$

Outline

Consider a general analytical function $u(\mu,\nu)+\mathrm{i}v(\mu,\nu)$ in the periodic half space $\mu\in[0,2]$, $\nu\in[-\infty,0]$ (i.e. infinite depth fluid). Using the mapping

$$\xi=\exp(-\mathrm{i}\pi(\mu+\mathrm{i}\nu)),$$

maps this domain to the inside of the unit circle.

Via Cauchy's integral theorem we have

$$u(\xi)+\mathrm{i}v(\xi)=-\frac{1}{2\pi\mathrm{i}}\oint_0^{2\pi}\frac{U(\theta)+\mathrm{i}V(\theta)}{\theta-\xi}\mathrm{d}\theta,$$

where $U(\mu)+\mathrm{i}V(\mu)=u(\mu,0)+\mathrm{i}v(\mu,0)$.

When evaluated on the edge of the disk, and using the above transformation leads to

$$U(\mu) + \mathrm{i}V(\mu) = \frac{1}{2}\int_0^2 (U(s) + \mathrm{i}V(s))\left[1 - \mathrm{i}\cot\left(\frac{\pi}{2}(\mu - s)\right)\right] \mathrm{d}s.$$

or

$$(\mathbf{I} + \mathrm{i}\mathbf{K})(U + \mathrm{i}V) = \overline{U} + \mathrm{i}\overline{V}\,.$$

Here

$$\mathbf{K}(U)(\mu) := \frac{1}{2}\int_0^2 U(s)\cot\left(\frac{\pi}{2}(\mu - s)\right) \mathrm{d}s\,,$$

is the Hilbert transform and

$$\overline{(\cdot)} = \frac{1}{2}\int_0^2 (\cdot)\, d\mu.$$

Taking real and imaginary parts shows how these quantities are linked on the free-surface

$$V = \overline{V} - \mathbf{K}(U) \quad \text{and} \quad U = \overline{U} + \mathbf{K}(V)\,.$$

Given V we then know U and vice versa.

In finite depth the bottom and surface conjugate functions are related (using the same procedure) by the Hilbert-Garrick transformation

$$\begin{bmatrix} \mathbf{I} + \mathrm{i}\big(\mathbf{K} + \mathbf{R}_q\big) & -\mathrm{i}\mathbf{S}_q \\ +\mathrm{i}\mathbf{S}_q & \mathbf{I} - \mathrm{i}\big(\mathbf{K} + \mathbf{R}_q\big) \end{bmatrix} \begin{pmatrix} U + \mathrm{i}V \\ U_b + \mathrm{i}V_b \end{pmatrix} = \begin{pmatrix} \overline{U} + \mathrm{i}\overline{V} \\ \overline{U}_b + \mathrm{i}\overline{V}_b \end{pmatrix},$$

where $U_b(\mu) + \mathrm{i}V_b(\mu) = u(\mu, -Q) + \mathrm{i}v(\mu, -Q)$.

Here $\mathbf{S}_q$ and $\mathbf{R}_q$ are integral transformations which depend upon the conformal modulus $Q(t)$.

Hence if we know (V, V_b) then we can use the Hilbert-Garrick transformation to determine (U, U_b) or vice versa.

Outline

Determining α

To find the value of α (the tangential fluid velocity at the surface) we note that z_t/z_ζ $(\zeta = \mu + \mathrm{i}\nu)$ is an analytic function with boundary values

$$\mathrm{Im}\left(\frac{z_t}{z_\mu}\right)\Bigg|^{\nu=0} = \frac{\Psi_\mu}{J} \quad \text{and} \quad \mathrm{Im}\left(\frac{z_t}{z_\mu}\right)\Bigg|_{\nu=-Q(t)} = 0.$$

Hence the real part of z_t/z_μ on the free-surface (which is just α/J) can be determined by the Hilbert-Garrick transformation as

$$\frac{\alpha}{J} = \overline{\alpha} + (\mathbf{K} + \mathbf{R}_q)\left(-\frac{\Psi_\mu}{J}\right) = \overline{\alpha} - \mathbf{T}_q^{-1}\left(\frac{\Psi_\mu}{J}\right).$$

Easy to show that $\overline{\alpha} = 0$.

Also, using the Hilbert-Garrick transformation we can relate

$$X - \mu = \mathbf{T}_q^{-1}(Y) \quad \text{and} \quad \Psi = \mathbf{T}_q(\Phi).$$

Therefore we integrate

$$\begin{aligned} Y_t &= -\frac{\Psi_\mu}{J} X_\mu - \mathbf{T}_q^{-1}\left(\frac{\Psi_\mu}{J}\right) Y_\mu, \\ \Phi_t &= -g(Y - h_0) - \frac{1}{2J}\Phi_\mu^2 + \frac{1}{2J}\Psi_\mu^2 - \mathbf{T}_q^{-1}\left(\frac{\Psi_\mu}{J}\right)\Phi_\mu, \end{aligned}$$

forward in time using 4th order Runge-Kutta, from some initial condition, for example if $F(t)$ is given then

$$X(\mu,0) = \mu, \quad Y(\mu,0) = \eta_0, \quad \Phi(\mu,0) = 0, \quad \Psi(\mu,0) = 0.$$

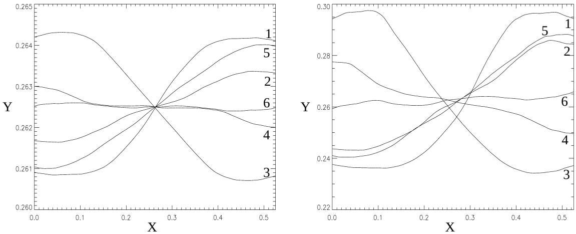
Discretize the free-surface via

$$\mu_k = (k-1)\frac{1}{N} \qquad k = 1, ..., 2N.$$

For this doubly-connected problem we can easily solve Laplace's equation for x, y, ϕ, and ψ, which give us the Fourier form of the Hilbert-Garrick operators, for example

$$\begin{aligned} x - \mu &= \sum_{n=1}^{\infty} A_n(t) \frac{\cosh((n\pi(\nu + Q)))}{\cosh(n\pi Q)} \sin(n\pi\mu), \\ y - \nu &= Q(t) + \sum_{n=1}^{\infty} A_n(t) \frac{\sinh((n\pi(\nu + Q)))}{\cosh(n\pi Q)} \cos(n\pi\mu). \end{aligned}$$

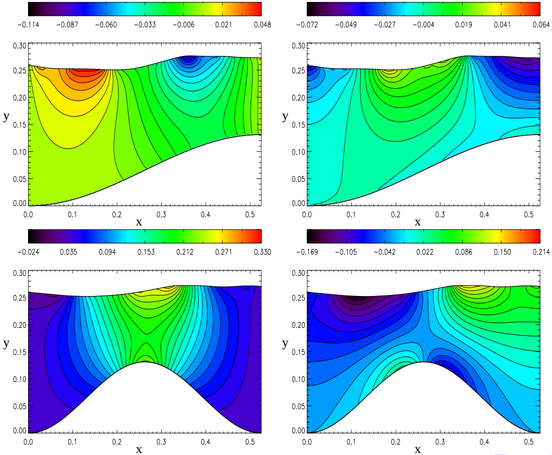
Hence we can use FFTs to determine the values of $A_n(t)$ on $\nu = 0$, reconstruct $X - \mu$ in Fourier space and then invert.



Forced variable bottom sloshing

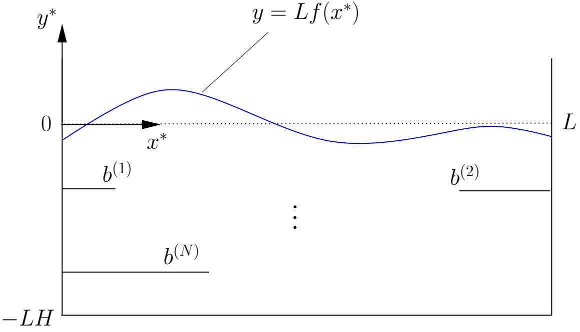
Forced rectilinear sloshing in rectangular tank with bottom topography. Harmonic forcing $F(t) = \epsilon \cos(\omega t)$.

Horizontal and vertical velocities:



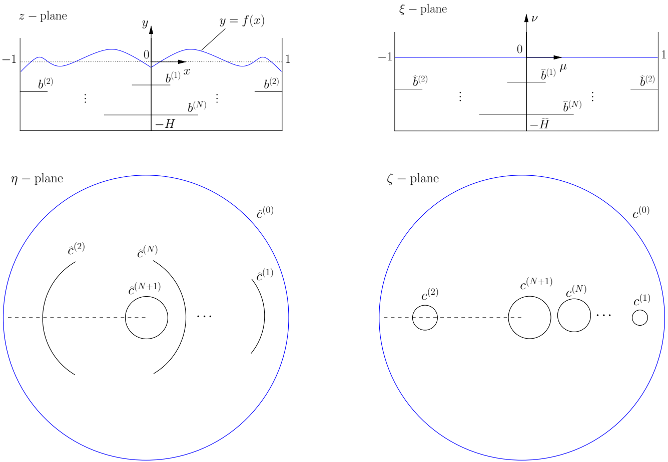
Outline

Can we apply a similar complex analysis approach to incorporate N infinitely thin side-wall baffles?



$$
\begin{aligned}
(x_0, y_0) &= (x(\mu, 0), y(\mu, 0)), \quad \mu \in [-1, 1], \\
(x_n, y_n) &= (x(\mu, -\widehat{y}_n), y(\mu, -\widehat{y}_n)), \quad \mu \in [-\widehat{L}_n, \widehat{L}_n], \\
&\quad \text{or} \quad \mu \in [-1, -1 + \widehat{L}_n] \cup [1 - \widehat{L}_n, 1], \\
(x_{N+1}, y_{N+1}) &= (x(\mu, -\widehat{H}), y(\mu, -\widehat{H})), \quad \mu \in [-1, 1].
\end{aligned}
$$

Again we map the domain



Final mapping uses Schottky-Klein prime functions via the MATLAB routines of Crowdy, Kropf, Green and Nasser.

In the mapped domain we again apply Cauchy's theorem

$$\kappa(\zeta) = -\frac{1}{2\pi\mathrm{i}} \oint_{\mathscr{C}} \frac{\kappa(\zeta')}{\zeta' - \zeta}\, \mathrm{d}\zeta',$$

where

$$\kappa(\zeta) = (x(\zeta) - \mu(\zeta)) + \mathrm{i}(y(\zeta) - \nu(\zeta)) = \widetilde{x}(\zeta) + \mathrm{i}\widetilde{y}(\zeta).$$

which can be expressed as

$$\begin{aligned}
\widetilde{x}(\zeta) + \mathrm{i}\widetilde{y}(\zeta) \quad = \quad & \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\widetilde{x}_0(\theta) + \mathrm{i}\widetilde{y}_0(\theta)}{e^{\mathrm{i}\theta} - \zeta} e^{\mathrm{i}\theta}\, \mathrm{d}\theta \\
& - \sum_{n=1}^{N} \frac{q_n}{2\pi} \int_0^{2\pi} \frac{\widetilde{x}_n(\phi) + \mathrm{i}\widetilde{y}_n(\phi)}{\delta_n + q_n e^{\mathrm{i}\phi} - \zeta} e^{\mathrm{i}\phi}\, \mathrm{d}\phi \\
& - \frac{q_{N+1}}{2\pi} \int_{-\pi}^{\pi} \frac{\widetilde{x}_{N+1}(\theta) + \mathrm{i}\widetilde{y}_{N+1}(\theta)}{\delta_{N+1} + q_{N+1} e^{\mathrm{i}\theta} - \zeta} e^{\mathrm{i}\theta}\, \mathrm{d}\theta.
\end{aligned}$$

This is then evaluated on the free-surface, the bottom and on each baffle to generate $N+2$ complex equations or $2N+4$ real equations. For example

$$
\begin{aligned}
\widetilde{x}_0(\sigma) + \mathrm{i}\widetilde{y}_0(\sigma) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\widetilde{x}_0(\theta) + \mathrm{i}\widetilde{y}_0(\theta)\right)\,\mathrm{d}\theta \\
&\quad - \frac{\mathrm{i}}{2\pi}\mathscr{P}\mathscr{V}\int_{-\pi}^{\pi} \cot\left[\frac{1}{2}(\theta-\sigma)\right]\left(\widetilde{x}_0(\theta) + \mathrm{i}\widetilde{y}_0(\theta)\right)\mathrm{d}\theta \\
&\quad - \sum_{n=1}^{N}\frac{q_n}{\pi}\int_{-\pi}^{\pi} F_{n0}(\sigma,\phi)(\widetilde{x}_n(\phi) + \mathrm{i}\widetilde{y}_n(\phi))\,\mathrm{d}\phi \\
&\quad - \frac{q_{N+1}}{\pi}\int_{-\pi}^{\pi} F_{(N+1)0}(\sigma,\theta)(\widetilde{x}_{N+1}(\theta) + \mathrm{i}\widetilde{y}_{N+1}(\theta))\,\mathrm{d}\theta.
\end{aligned}
$$

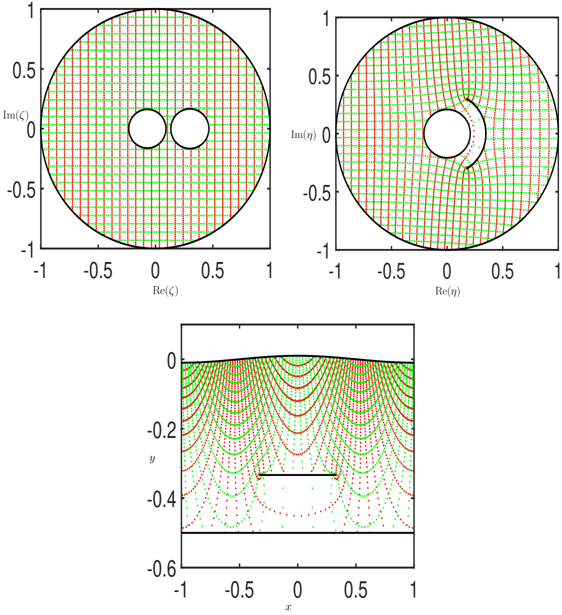
The 'F' functions are known functions in terms of the circle conformal moduli.

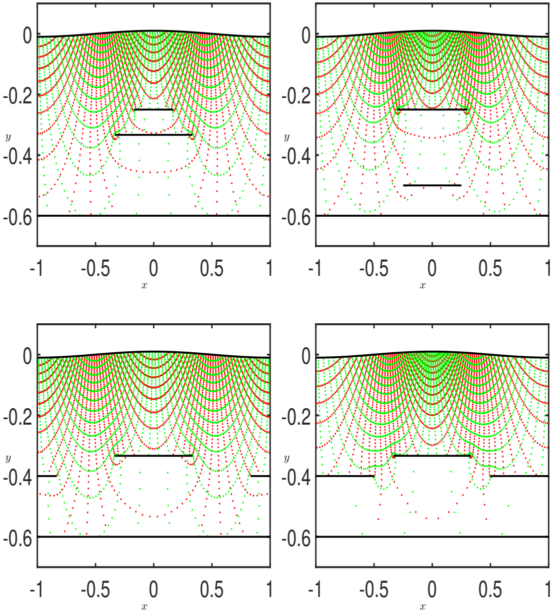
In order to perform a feasibility study for using this complex analysis approach for the dynamic fluid problem, we first consider a kinematic problem, such that the free-surface is specified, i.e $\eta(x,t) = f(x) = a\cos(\pi x)$.

In this case it means that we know $\widetilde{y}_0,\ \widetilde{y}_1,\ \widetilde{y}_2, ..., \widetilde{y}_{N+1}$ and we need to calculate the corresponding $\widetilde{x}_n$ values.

We discretize the integrals with equally spaced grid points, and then evaluate the equations from Cauchy's theorem at the mid-point of these grid points. This the PV integrals can be evaluated via the Trapezoidal rule.

These equations are then evaluated and we iteratively update the values of $\widehat{y}_n$, $\widehat{L}_n$ and $\widehat{H}$. Essentially this is fixing the conformal moduli. ($\widehat{H}(t)$ equivalent to $Q(t)$ in earlier notation)


Im(ζ)
Re(ζ)
Im(η)
Re(η)
y
x
1
0.5
0
-0.5
-1
-0.2
-0.4
-0.6


0
-0.2
-0.4
-0.6
y
-1
-0.5
0
0.5
1
x

Outline

This feasibility study shows that the dynamic problem should be obtainable.

Some issues arising are:

- Equal spacing in μ does not translate to equal spacing in θ etc. So interpolation is needed at each iteration. Will this cause issues?

For the dynamic coupled problem $F(t)$ satisfies an equation similar to

$$(m_v + m_f)\ddot{F} + \nu F = -\frac{d}{dt}\int_0^L \int_0^{\eta(x,t)} \frac{\partial \phi}{\partial x}\, dy\, dx,$$

hence integral needs to be evaluated. But using Green's theorem this appears to be just line integrals along the surface, bottom and baffles.

Thank you