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On Energy Conservation for the hydrostatic Euler equations: an Onsager Conjecture

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Introduction

Topics of this presentation:

- Anamolous dissipation and Onsager's conjecture
- (Inviscid) primitive equations of oceanic and atmospheric dynamics
- Sufficient conditions for energy conservation of weak solutions to the hydrostatic Euler equations
- New type of weak solution using harmonic analysis

Description of an inviscid fluid

One of the main models of fluid mechanics are the incompressible Euler equations

$$\partial_t u + (u \cdot \nabla) u + \nabla p = 0, \quad \nabla \cdot u = 0.$$

Classical solutions of these equations conserve the kinetic energy (by an elementary computation), which is given by

$$\frac{1}{2} \int_{\mathbb{T}^3} |u|^2 dx.$$

What happens for weak solutions?

Onsager's conjecture

The conjecture formulated in [Onsager, 1949] states that:

- If $u(\cdot, t) \in C^{0,\theta}$ with $\theta > \frac{1}{3}$, the solution converges energy.
- If $u(\cdot, t) \in C^{0,\theta}$ with $\theta < \frac{1}{3}$, there exist solutions that dissipate energy (not necessarily all solutions).

Therefore $\theta = \frac{1}{3}$ is known as the Onsager exponent. It is the threshold for energy conservation. Here a regularity condition is tied to a physical phenomenon.

In particular, to model anomalous dissipation, weak solutions are necessary.

Lars Onsager

In 1949, Lars Onsager published the paper 'Statistical Hydrodynamics'. In it he introduced two very important concepts in hydrodynamical turbulence:

- The concept of anolamous dissipation, which states that there is a mechanism for energy dissipation independent of viscosity.
- A theory for the spontaneous formation of large scale, long lived vortices in 2D fluids.

We will only discuss the former. It discusses a very fundamental fact of the Euler equations. The viscosity is not the only source of energy dissipation.

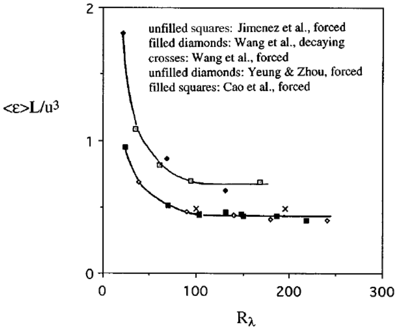
Onsager's paper

"It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such "ideal" turbulence cannot obey any LIPSCHITZ condition of the form

$$|v(r' + r) - v(r')| \leq (const.) r^n,$$

for any order n greater than 1/3; otherwise the energy is conserved."

Experimental and numerical verification



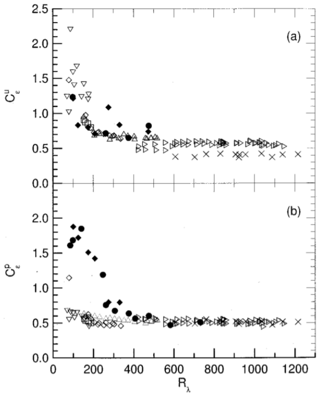


Figure: Verification of anomalous dissipation, sources: [Sreenivasan, 1998; Pearson, Krogstad, and van de Water, 2002]

Approaches to prove energy conservation

Theorem (Constantin-E-Titi 1994)

If $u \in L^3((0,T); B^{\theta}_{3,\infty}(\mathbb{T}^3))$ with $\theta > \frac{1}{3}$ is a weak solution of the Euler equations, it must conserve the energy.

- Commutator estimates, see [Constantin, E, and Titi, 1994]. One derives a global energy conservation equation, with additional ‘source’ terms which are zero under sufficient regularity using commutator estimates.
- Energy equation, see [Duchon and Robert, 2000]. One derive an equation describing the local energy balance, which contains a term that captures the lack of regularity. This is roughly the approach Onsager used in 1949.

Convex integration

Theorem (Isett 2018; Buckmaster-De Lellis-Székelyhidi-Vicol 2019)

Let $e : [0, T] \to \mathbb{R}_+$ *be any smooth nonnegative function. Then for any* $\theta < \frac{1}{3}$ *there exists a solution* $u \in C^\theta((0, T) \times \mathbb{T}^3)$ *of the Euler equations such that*

$$\int_{\mathbb{T}^3} |u(x, t)|^2 dx = e(t).$$

Previous work on the Onsager conjecture

- First half of Onsager conjecture: [Eyink, 1994; Constantin, E, and Titi, 1994; Duchon and Robert, 2000; Cheskidov, Constantin, Friedlander, and Shvydkoy, 2008; Cheskidov, Lopes Filho, Nussenzveig Lopes, and Shvydkoy, 2016; Robinson, Rodrigo, and Skipper, 2018; Bardos, Gwiazda, Świerczewska-Gwiazda, Titi, and Wiedemann, 2019]
- Second half of Onsager conjecture: [De Lellis and Székelyhidi, 2009, 2010; Buckmaster, De Lellis, Isett, and Székelyhidi, 2015; Daneri and Székelyhidi, 2017; Isett, 2018; Buckmaster, De Lellis, Székelyhidi, and Vicol, 2018]

Primitive equations

The primitive equations are given by

$$
\begin{aligned}
&\partial_t v + (v \cdot \nabla) v + w \partial_z v + \nabla p + \Omega v^{\perp} - \nu_h \Delta v - \nu_z \partial_{zz} v = 0, \\
&\partial_z p + T = 0, \\
&\nabla \cdot v + \partial_z w = 0, \\
&\partial_t T + v \cdot \nabla T + w \partial_z T - \kappa_h \Delta T - \kappa_z \partial_{zz} T = Q.
\end{aligned}
$$

They arise from the small aspect ratio, see [Azérad and Guillén, 2001; Li and Titi, 2019].

Previous work on the primitive equations I

For the viscous primitive equations:

- Derivation: [Richardson, 1922; Lions, Temam, and Wang, 1992]
- Short-time existence: [Guillén-González, Masmoudi, and Rodríguez-Bellido, 2001]
- Global existence: [Cao and Titi, 2007; Kobelkov, 2006; Kukavica and Ziane, 2007; Hieber and Kashiwabara, 2016]
- Small-aspect ratio: [Azérad and Guillén, 2001; Bresch, Guillén González, Masmoudi, and Rodríguez Bellido, 2001; Li and Titi, 2019]

For the inviscid primitive equations:

- Ill-posedness in Sobolev spaces: [Renardy, 2009; Han-Kwan and Nguyen, 2016]

Previous work on the primitive equations II

- Finite-time singularity: [Cao, Ibrahim, Nakanishi, and Titi, 2015; Wong, 2015]
- Local well-posedness for analytic data: [Kukavica, Temam, Vicol, and Ziane, 2011; Gerard-Varet, Masmoudi, and Vicol, 2020]
- With rotation: [Ibrahim, Lin, and Titi, 2021; Ghoul, Ibrahim, Lin, and Titi, 2022]
- Nonuniqueness of weak solutions for the inviscid case: [Feireisl, 2016; Chiodaroli and Michálek, 2017]

Hydrostatic Euler

The hydrostatic Euler equations (inviscid Primitive equations of Oceanic and Atmospheric Dynamics) are given by

$$\begin{aligned} &\partial_t \mathbf{u}_h + (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h + w\partial_z \mathbf{u}_h + \nabla p = 0, \\ &\nabla \cdot \mathbf{u}_h + \partial_z w = 0, \quad \partial_z p = 0. \end{aligned}$$

∇ is 2-dimensional and w is no longer an independent quantity. The formally conserved quantity is $\|u\|_{L^2}^2 + \|v\|_{L^2}^2$.

We looked at an analogue of Onsager's conjecture for the hydrostatic Euler equations. We want to find sufficient conditions for a weak solution of the equations to conserve energy.

Main differences with the Euler equations

The equation for w can be written as

$$w = -\int_0^z \left(\partial_x u + \partial_y v\right) dz'.$$

Main features:

- Nonlocality
- Anisotropy in regularity

What to do with w?

- Fix a regularity assumption on w
- Fix sufficiently strong assumptions on u and v, without explicit assumptions on w

Approaches to prove energy conservation

Approaches to formulate Onsager-type conjectures:

- Use anisotropic conditions
- Weaken regularity requirements on w (make it a functional)
- Make no assumptions on w

Two different notions of weak solutions

- Assume that $w \in L^2_t(L^2_x)$ and $u, v \in L^\infty_t(L^2_x)$ (weak solution)
- Assume that $w \in L^2_t(B^{-s}_{2,\infty})$ and $u, v \in L^4_t(B^{s+}_{4,2})$ for $0 < s < \frac{1}{2}$ (very weak solution)

Result for weak solutions

We first have the following 'classical' Onsager-type conjecture.

Theorem (DB-Markfelder-Titi 2022)

Let $\mathbf{u}$ *be a weak solution of the hydrostatic Euler equations with* $u, v \in L_t^\infty(L_x^2)$ *and* $w \in L_t^2(L_x^2)$*, then if* $u, v \in L_t^4(B_x^{1/2+})$ *the solution conserves energy.*

We assume that $w(\cdot, t) \in L^2$, this implicitly means that $\partial_x u + \partial_y v(\cdot, t) \in L^2$ by the equation

$$w = -\int_0^z \left(\partial_x u + \partial_y v\right) dz'.$$

Anisotropic conditions

One can also make use of the anisotropy of the equation to obtain the following result:

Theorem (DB-Markfelder-Titi 2022)

Let u, v have Besov regularity $B^{\alpha}_{3,\infty}$ in the z-direction, and regularity $B^{\beta}_{3,\infty}$ in the horizontal directions, then the solution conserves energy if

$$\alpha > \frac{1}{3}, \quad \beta > \frac{2}{3}, \quad \beta + 2\alpha > 2.$$

So the Onsager exponent in the z-direction can be 'lowered' to $\frac{1}{3}$ at the expense of a higher exponent in the horizontal directions.

Definition of very weak solution

We introduce a new notion of weak solution of the hydrostatic Euler equations.

Definition

Let $u, v \in L_t^\infty(L_x^2) \cap L_t^4(B_{4,2}^{s+})$ and $w \in L_t^2(B_{2,\infty}^{-s})$ for fixed $0 < s < \frac{1}{2}$.
A very weak solution satisfies (with a similar equation for v),

$$\int_0^T \int_{\mathbb{T}^3} u\partial_t\phi_1 dx\ dt + \int_0^T \langle u\mathbf{u}, \nabla\phi_1\rangle dt + \int_0^T \int_{\mathbb{T}^3} \Omega v\phi_1 dx\ dt$$
$$+ \int_0^T \int_{\mathbb{T}^3} p\partial_x\phi_1 dx\ dt = 0.$$

How to make sense of *uw* and *vw* as a distribution?

Littlewood-Paley theory

Littlewood-Paley theory is about localisation in frequency space. Let $\chi \in \mathcal{D}(B(0,4/3))$ and $\varphi \in \mathcal{D}(A(3/4,8/3))$ such that we have the partition of unity.

$$\chi(\xi) + \sum_{j=0}^{\infty} \varphi(2^{-j}\xi) = 1.$$

We introduce the dyadic blocks as follows

$$\widehat{\Delta_j f} = \varphi_j \widehat{f}.$$

Paradifferential Calculus

Bony's decomposition of a product is given by

$$fg = T_f g + T_g f + R(f,g).$$

Here $T_f g$ and $T_g f$ are called the paraproducts which are given by

$$T_f g = \sum_{j=-1}^{\infty} \sum_{i=-1}^{j-2} \Delta_i f \Delta_j g, \quad T_g f = \sum_{j=-1}^{\infty} \sum_{i=-1}^{j-2} \Delta_i g \Delta_j f.$$

These are the 'low-high' and 'high-low' frequencies. The resonance term $R(f,g)$ is given by

$$R(f,g) = \sum_{|k-j|\leq 1} \Delta_k f \Delta_j g.$$

These are the 'high-high' frequencies. $T_f g$ and $T_g f$ always lie in some Besov space, but the trouble might be in the resonance term.

Paraproduct estimates

Lemma (Paraproduct estimates)

If $f \in L^{p_1}$ and $g \in B^{\beta}_{p_2,q}$ then

$$\|T_f(g)\|_{B^{\beta}_{p,q}} \leq C\|f\|_{L^{p_1}}\|g\|_{B^{\beta}_{p_2,q}}.$$

If $f \in B^{\alpha}_{p_1,q_1}, g \in B^{\beta}_{p_2,q_2}$ with $\alpha < 0$ and $\beta \in \mathbb{R}$

$$\|T_f(g)\|_{B^{\alpha+\beta}_{p,q}} \leq (1-2^{\alpha})^{-1}\|f\|_{B^{\alpha}_{p_1,q_1}}\|g\|_{B^{\beta}_{p_2,q_2}}.$$

Note that we assumed that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \min\left\{1, \frac{1}{q_1} + \frac{1}{q_2}\right\}.$$

Resonance estimates

Lemma (Resonance estimates)

Assume that $f \in B^{s_1}_{p_1,q_1}$ *and* $g \in B^{s_2}_{p_2,q_2}$ *such that* $s_1 + s_2 > 0$*, then*

$$\|R(f,g)\|_{B^{s_1+s_2}_{p,q}} \le \|f\|_{B^{s_1}_{p_1,q_1}} \|g\|_{B^{s_2}_{p_2,q_2}},$$

under the assumption that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \le 1, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \le 1.$$

Sufficient condition for very weak solutions

By applying the paradifferential calculus we obtain the next lemma.

Lemma

Let $0 < s < \frac{1}{2}$, $w(\cdot, t) \in B^{-s}_{2,\infty}$ *and* $u(\cdot, t), v(\cdot, t) \in B^{s+}_{4,2}$, *then* $uw(\cdot, t), vw(\cdot, t) \in B^{-s}_{1,\infty}$.

Theorem (DB-Markfelder-Titi 2022)

Let $(\mathbf{u}, s)$ *be a very weak solution of the hydrostatic Euler equations, then if* $u, v \in L^4_t(B^{s+1/2+}_{4,\infty})$, *energy is conserved.*

Moreover, if $u, v \in L^4_t(B^{3/4+}_{4,\infty})$ *(with no conditions on* w*) then energy is conserved.*

So the Onsager exponent increases with s. We now have an Onsager ‘scale’.

Summary and further results

Theorem (DB-Markfelder-Titi 2022)

Energy is conserved under any of the following conditions.

- *If* $u, v \in L^4_t(B^{1/2+}_{4,\infty})$ *and* $w \in L^2_t(L^2_x)$*, also* $u, v \in L^3_t(C^{1/2+}_{\log})$ *suffices*
- *If* $w \in L^3_t(C^\beta_x)$ *and* $u, v \in L^3_t(C^\alpha_x)$ *with* $\alpha > \frac{1}{2} - \frac{1}{2}\beta$
- *If* $w \in L^2_t(L^2_x)$ *and* u *and* v *have Besov regularity* $B^\alpha_{3,\infty}$ *vertically and* $B^\beta_{3,\infty}$ *horizontally if* $\alpha > \frac{1}{3}, \beta > \frac{2}{3}$ *and* $\beta + 2\alpha > 2$
- *If* $u, v \in L^3_t(W^{1,p}_x \cap C^\beta_x)$ *with* $\beta > \frac{1}{2}$ *and* $p > 1$ *or* $u, v \in L^3_t(W^{1,p}_x)$ *with* $p > 6$
- *If* $u, v \in L^3_t(B^{1+}_{9/4,\infty})$

Conclusion

- The hydrostatic Euler equations have a ‘family’ of Onsager conjectures
- This is due to the anisotropic regularity and nonlocal nature of the velocity field
- The different approaches can probably be combined

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