The soliton vs the gas

Fredholm determinants, analysis, and finer details

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joint with Tamara Grava (SISSA-Bristol), Bob Jenkins (UCF), Ken T-R McLaughlin (CSU),
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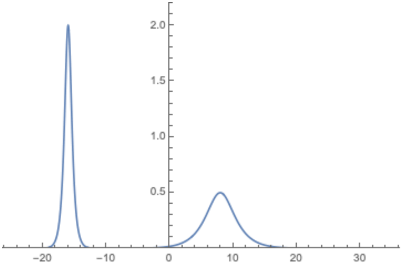
The mighty "Wave of Translation"

The (focusing) modified KdV equation

$$q_t + 6q^2 q_x + q_{xxx} = 0$$

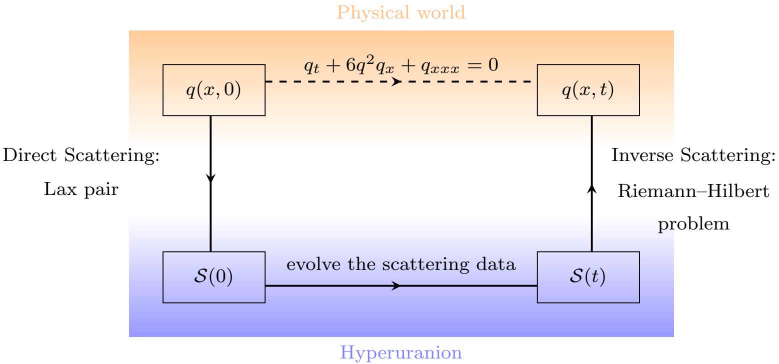
is a nonlinear *integrable* PDE, arising as the compatibility condition of a Lax pair of linear differential operators (Lax, '68):

$$\mathbf{\Phi}_x = \begin{pmatrix} -\mathrm{i}k & q(x,t) \\ -q(x,t) & \mathrm{i}k \end{pmatrix} \mathbf{\Phi},$$
$$\mathbf{\Phi}_t = \begin{pmatrix} -4\mathrm{i}k^3 + 2\mathrm{i}kq^2 & 4k^2 q + 2\mathrm{i}kq_x - 2q^3 - q_{xx} \\ -4k^2 q + 2\mathrm{i}kq_x + 2q^3 + q_{xx} & 4\mathrm{i}k^3 - 2\mathrm{i}kq^2 \end{pmatrix} \mathbf{\Phi}\ .$$



The general solution

Recipe (for fast decaying or step-like IC):



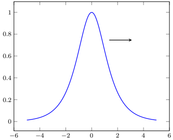
The scattering data are the L^2-eigenvalues $\{-i\kappa_j\}$, the norming constants $\{\chi_j\}$, and the reflection coefficient $r(k)$.

Classical solutions

1. rapidly decreasing, localized travelling wave (*soliton*):
   $q_{\mathrm{sol}}(x,t) = 2\,\mathrm{sgn}(\chi)\kappa \ \mathrm{sech}\left(2\kappa(x-4\kappa^2 t) + x_0\right)$

where iκ is the discrete eigenvalue and the norming constant χ is related to the phase shift x_0 by

$$x_0 = \log \frac{2\kappa}{|\chi|} \in \mathbb{R}.$$



The solution with $\chi > 0$ coincides with the soliton, while $\chi < 0$ gives the anti-soliton.

2. *periodic travelling wave* solutions:

$$q_{\mathrm{ell}}(x,t) = a\,\mathrm{dn}\left(a_1(x - Vt + x_0) \mid m\right)$$

where $\mathrm{dn}\,(z \mid m)$ is Jacobi elliptic function of modulus $m \in (0,1)$.

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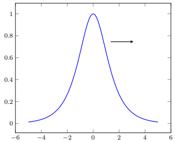

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What is a soliton gas?

Definition

A **soliton gas/ensemble** is a (random) configuration of a large number of solitons.

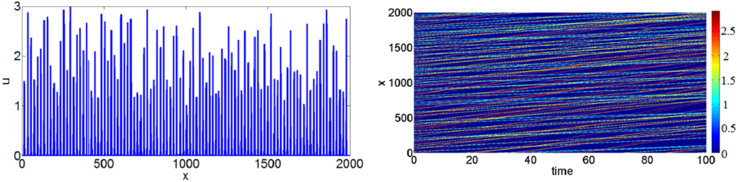
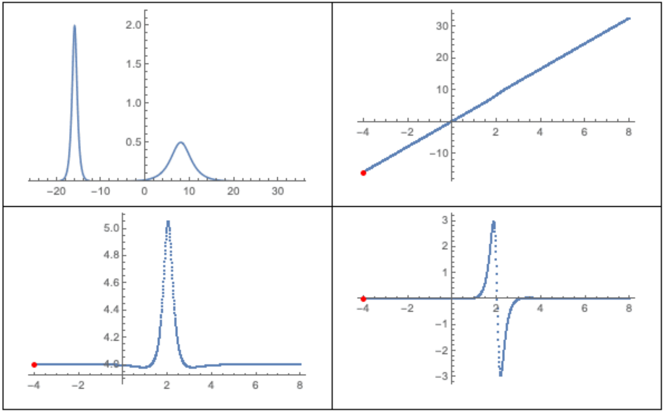


Figure: Initial distribution of the soliton gas (left). (t, x)-diagram of soliton field (right). From [Shurgalina, Pelinovski, '16].

The original gas [Zakharov, '71]

Assume the soliton gas to be *dilute* with density $\rho(\kappa; x, t)$.
Launch a "test" soliton $\Rightarrow$ isolated pairwise interactions, positions shifts are effectively altering its velocity



In such a configuration the velocity of the trial soliton satisfies the kinetic+continuity equations

$$v(\kappa) = 4\kappa^2 + \frac{4}{\kappa}\int_0^\infty \ln\left|\frac{s+\kappa}{s-\kappa}\right| \left(\kappa^2 - s^2\right)\rho(s;x,t)\,\mathrm{d}s$$

$$\rho_t + (v\rho)_x = 0$$

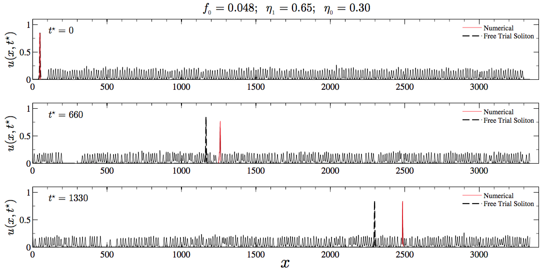


Figure: From [Carbone, Dutyk, El, '16].

More history

- In 2003, El derived an integral equation for the velocity v of the trial soliton with point spectrum κ propagating in a *dense* soliton gas:

$$\boxed{\begin{aligned} &v(\kappa) = 4\kappa^2 + \frac{1}{\kappa}\int_0^\infty \ln\left|\frac{s+\kappa}{s-\kappa}\right| (v(\kappa) - v(s))\, \rho(s; x, t)\, \mathrm{d}s \\ &\rho_t + (v\rho)_x = 0 \end{aligned}}$$

Extended in several directions, by [El, Kamchatnov, '05], [Pelinovsky, Dutykh, '14], [Pelinovsky, Shurgalina, '17], [El,Tovbis, '20], and others...

- [Zaitsev, Whitham, '83] and [Boyd, '84] showed that the sum infinite solitons

$$q(x) = \sum_{n=-\infty}^{\infty} \operatorname{sech}^2(x - n\sigma)$$

coincides with the elliptic solution of KdV.

- [Gesztesy, Karwowski, Zhao, '92] showed the existence of the infinite soliton limit for KdV when the point spectrum $\kappa_j \in \ell^\infty(\mathbb{N})$ and the norming constants $c_j^2/\kappa_j \in \ell^1(\mathbb{N})$.

- The first instance of a *dense* (KdV) soliton gas can be found in [Dyachenko, Zakharov, Zakharov, '16], where the gas is constructed via dressing method (*primitive potential*).
These potentials are constructed via a scalar *nonlocal* RH problem.

  Subsequent works by [Nabelek, Zakharov, Zakharov, '20–'21].

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Soliton gas as *continuum limit* of N-soliton solution

The N-solitons solution [Wadati, '72]:

$$\operatorname*{res}_{k=\mathrm{i}\kappa_j} \boldsymbol{M} = \lim_{k\to\mathrm{i}\kappa_j} \boldsymbol{M}(k)\begin{bmatrix} 0 & 0 \\ -\mathrm{i}\chi_j\,\mathrm{e}^{-2\mathrm{i}\theta(k;x,t)} & 0 \end{bmatrix}$$

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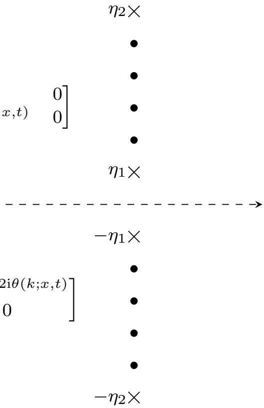


with $\theta(k;x,t) := 4tk^3 + xt$. Then,

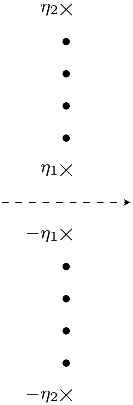
$$q(x,t) = 2\mathrm{i}\lim_{k\to\infty} k\boldsymbol{M}(k;x,t)_{12}$$

But also

$$q(x,t) = \mathrm{i}\frac{\partial}{\partial x}\ln\det\left(\boldsymbol{I}_N - \mathrm{i}\boldsymbol{A}\right) - \mathrm{i}\frac{\partial}{\partial x}\ln\det\left(\boldsymbol{I}_N + \mathrm{i}\boldsymbol{A}\right)$$

with $\boldsymbol{A} \in \mathbb{R}^{N\times N}$,

$$\boldsymbol{A}_{j\ell} = \frac{\sqrt{\chi_j}\sqrt{\chi_\ell}}{\kappa_j + \kappa_\ell} e^{-\mathrm{i}\left[\theta(\mathrm{i}\kappa_j;x,t)+\theta(\mathrm{i}\kappa_\ell;x,t)\right]}$$



Assume poles are accumulating uniformly in $\Sigma_1 \cup \Sigma_2$:

- Rescale $\chi_j \mapsto \frac{\chi_j}{N}$
- The norming constants χ_j are discretization of a smooth function $r(k)$ (real-valued, positive)
- Take the limit $N \to +\infty$ ...

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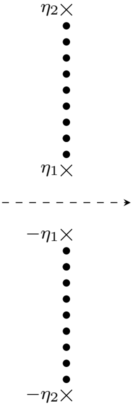

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The soliton gas potential

Theorem (GGJMcLM, ’22; part I)

The Fredholm determinant expression

$$q(x,t) = \mathrm{i}\frac{\partial}{\partial x}\ln\det\left(\mathrm{Id}_{L^2(\Sigma_1)} + \mathcal{K}\right) - \mathrm{i}\frac{\partial}{\partial x}\ln\det\left(\mathrm{Id}_{L^2(\Sigma_1)} - \mathcal{K}\right),$$

is the soliton gas solution of the mKdV equation, where the integral operator $\mathcal{K}$ has kernel

$$K(k,z) = \frac{\sqrt{r(k)}e^{-\mathrm{i}\,\theta(k;x,t)}\,\sqrt{r(z)}e^{-\mathrm{i}\,\theta(z;x,t)}}{2\pi\mathrm{i}(k+z)}, \qquad k,z\in\Sigma_1\ .$$

- Similar formulæ hold for the soliton+gas solution of the (m)KdV [GGJMcLM, ’22], [GGJMcL, ’21].
- An equivalent solution was obtained in [Tracy, Widom, ’96] for the defocusing mKDV.
- Similar solutions have emerged in relation to defocusing mKDV in the study of fluctuations in random matrices and KPZ universality classes [Baik, Bothner, ’20], [Krajenbrink, Le Doussal, ’21].

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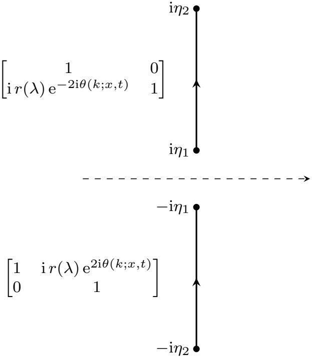
Using techniques from [Bertola, Cafasso, '12] we can also get a Riemann–Hilbert problem.

Theorem (part II)

The mKdV soliton gas solution can be equivalently recovered as

$$q(x,t) = 2\mathrm{i} \lim_{k\to\infty} k\boldsymbol{X}(k;x,t)_{12} \ .$$

- this is really a soliton *gas*!
  (the long-time asymptotics produces densities satisfying El–Zakharov's kinetic equation)
- this gas is *regular*
- and *dense*
  (see [Dyachenko, Zakharov, Zakharov, '16])



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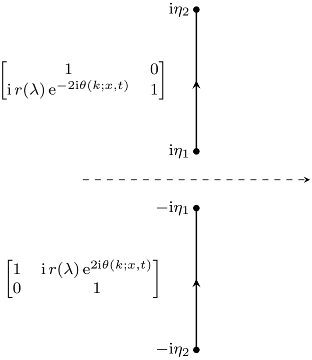

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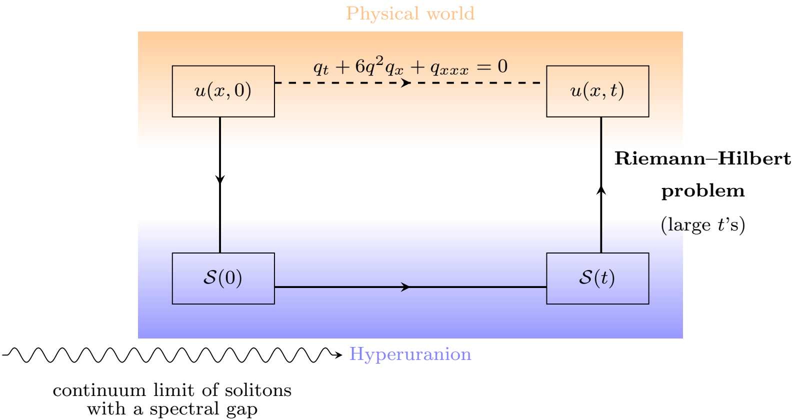
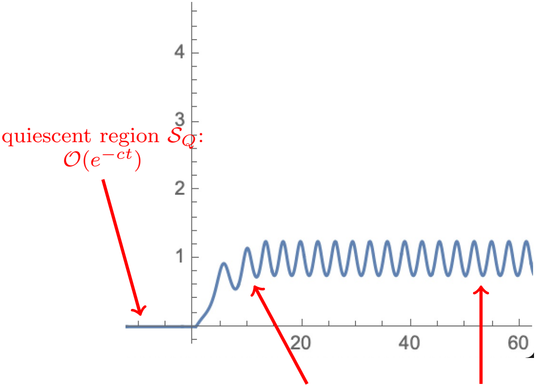
Asymptotic analysis



The (free) soliton gas ([GGJMcL, '21] and [GGJMcLM, '22])

- $q(x,0)$ is step-like: $q(x,0) = (\eta_2+\eta_1)\mathrm{dn}\left((\eta_2+\eta_1)(x-x_0), m\right) + \mathcal{O}\left(x^{-1}\right)$ as $x \to +\infty$ and $q(x,0) = O(e^{-cx})$ as $x \to -\infty$.
- $q(x,t)$ is asymptotically a *rarefaction wave* as $t \to +\infty$.



$$q_{\mathrm{ell}}(x,t) = (\alpha+\eta_1)\mathrm{dn}\left((\alpha+\eta_1)(x-2(\eta_1^2+\alpha^2)t-x_0), m_\alpha\right) + \mathcal{O}(t^{-1})$$

The phase in the jumps

$$4k^3t + kx = 4tk\left(k^2 + \frac{x}{4t}\right)$$

- for $\frac{x}{t} < 4\eta_1^2$, $q(x,t)$ is exponentially small.
- $4\eta_1^2 < \frac{x}{t} < v_{\mathrm{crit}}$: $q(x,t)$ is a periodic travelling wave with slowly varying parameters; $\alpha = \alpha(x/t)$ is the self-similar solution of the Whitham modulation equations.
- $\frac{x}{t} > v_{\mathrm{crit}}$: $q(x,t)$ is a periodic travelling wave with fixed parameters ($\alpha \equiv \eta_2$).

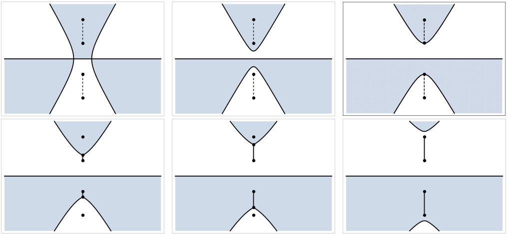


Figure: Zero levels of $\Im(4tk^3 + kx)$ for different values of $\frac{x}{t}$.

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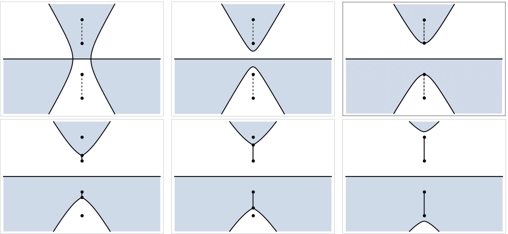

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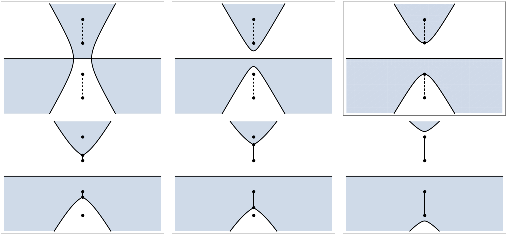

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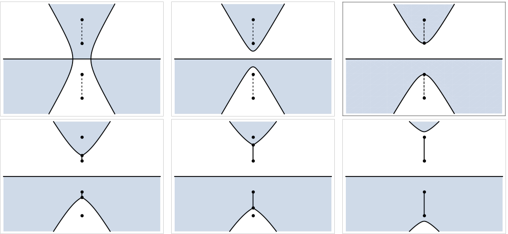


Figure: Zero levels of $\Im(4tk^3 + kx)$ for different values of $\frac{x}{t}$.

The DZ steepest descent business

Massage the RH problem [Deift, Zhou, '92]:

$$\boldsymbol{T}(k) = \boldsymbol{X}(k) e^{-\mathrm{i}g(k)\sigma_3} f(k)^{\sigma_3}$$

The dynamic is driven by the g-function

$$g(k;x,t) = \int_{\Sigma_\alpha \cup \overline{\Sigma}_\alpha} \log(k-s)\rho(s;x,t)\,\mathrm{d}s\ , \quad \Sigma_\alpha = [\mathrm{i}\eta_1, \mathrm{i}\alpha]$$

and the *wave phase* $\varphi(k;x,t) := g(k;x,t) + kx + 4k^3 t$.

The measure $\rho(s)\,\mathrm{d}s$ is given explicitly

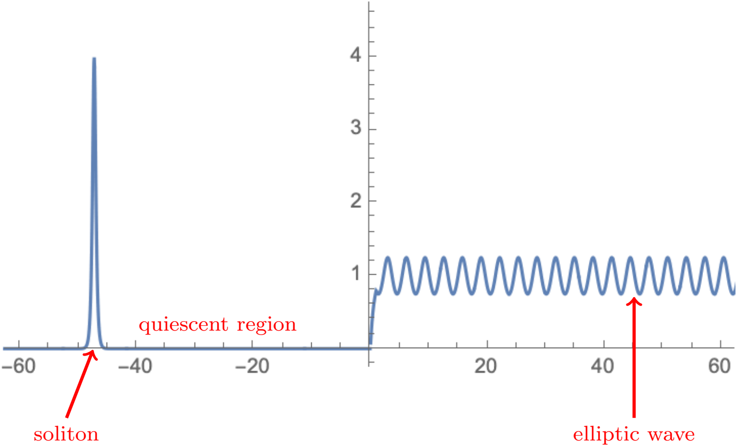
$$\rho(s;x,t)\mathrm{d}s = -\frac{1}{\pi \mathrm{i}} \frac{12t(s^4 + \frac{1}{2}(\eta_1^2+\alpha^2)s^2 + c_2) + x(s^2+c_0)}{\sqrt{(s^2+\alpha^2)(s^2+\eta_1^2)}} \mathrm{d}s$$

for some constants c_0, c_2 depending on x and t, uniquely determined by

$$\int_{-i\eta_1}^{i\eta_1} \rho(s;x,t)\mathrm{d}s = 0.$$

The soliton VS the gas

Add a *large* ($\kappa_0 > \eta_2$) soliton, initialized on the *left* of the gas ($x_0 \ll -1$):



The RH problem

$$\operatorname*{res}_{k=\mathrm{i}\kappa_0} \boldsymbol{X} = \lim_{k\to \mathrm{i}\kappa_0} \boldsymbol{X}(k) \begin{bmatrix} 0 & 0 \\ -\mathrm{i}\chi\, \mathrm{e}^{-2\mathrm{i}\theta(k;x,t)} & 0 \end{bmatrix}$$

$$\boldsymbol{X}_+(k) = \boldsymbol{X}_-(k) \begin{bmatrix} 1 & 0 \\ \mathrm{i}\, r(\lambda)\, \mathrm{e}^{-2\mathrm{i}\theta(k;x,t)} & 1 \end{bmatrix}$$

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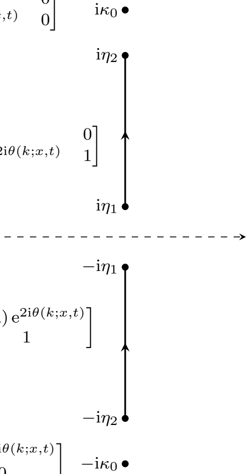
$$\operatorname*{res}_{k=-\mathrm{i}\kappa_0} \boldsymbol{X} = \lim_{k\to -\mathrm{i}\kappa_0} \boldsymbol{X}(k) \begin{bmatrix} 0 & -\mathrm{i}\chi\, \mathrm{e}^{2\mathrm{i}\theta(k;x,t)} \\ 0 & 0 \end{bmatrix}$$

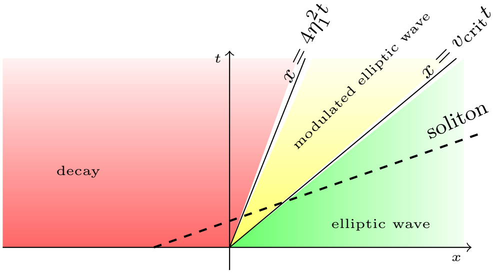


The cumbersome presence of the soliton

The structure of the g-function separates the (x,t) half-plane into the usual sectors $\mathcal{S}_Q \cup \mathcal{S}_M \cup \mathcal{S}_E$.
We further subdivide these sectors into

$$\mathcal{S}^{(\pm)} = \mathcal{S}_Q^{(\pm)} \cup \mathcal{S}_M^{(\pm)} \cup \mathcal{S}_E^{(\pm)},$$

- $\mathcal{S}^{(-)}$ = points behind the soliton or after the soliton has passed by (the residue condition at $\mathrm{i}\kappa_0$ is asymptotically small)
- $\mathcal{S}^{(+)}$ = point in front of the soliton or before the soliton passes by (the residue condition is asymptotically large $\rightsquigarrow$ reverse triangularity, **phase shift**...)

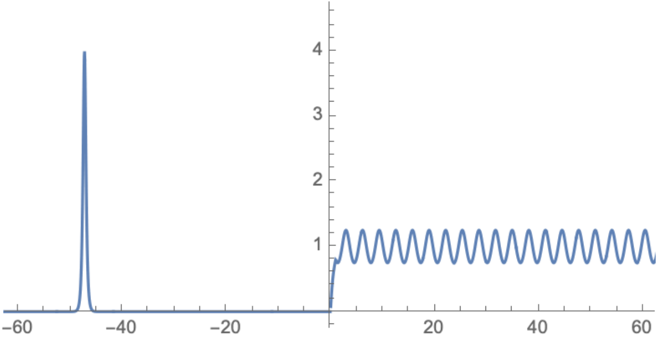


In the quiescent region

When $\xi < 4\eta_1^2$ (i.e. $(x,t) \in \mathcal{S}_Q$),

$$\boldsymbol{X}(k;x,t) = \left[\boldsymbol{I} + \mathcal{O}\left(e^{-2t\eta_1(4\eta_1^2-\xi)}\right)\right]\boldsymbol{X}^{(\mathrm{sol})}(k;x,t), \quad \text{as } t \to \infty$$

where $\boldsymbol{X}^{(\mathrm{sol})}(k;x,t)$ is the solution of the RH problem with $r \equiv 0$ and poles at $\pm\mathrm{i}\kappa_0$.

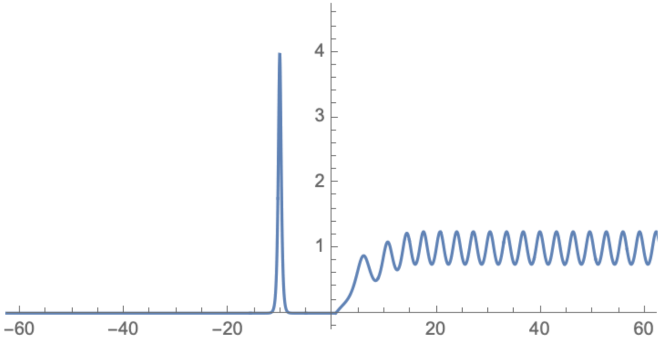


In the elliptic regions

As t grows, the soliton is getting trapped inside the soliton gas; the trailing edge of the soliton gas is propagating at a speed $x = 4\eta_1^2 t$.

For $4\eta_1^2 < \frac{x}{t} < v_{\text{crit}}$ (i.e. $(x,t) \in \mathcal{S}_M$) and $\frac{x}{t} > v_{\text{crit}}$ (i.e. $(x,t) \in \mathcal{S}_E$) , introduce the g-function, open lenses and solve a model problem.

Take into account the *highly non-trivial* interaction soliton vs gas ...



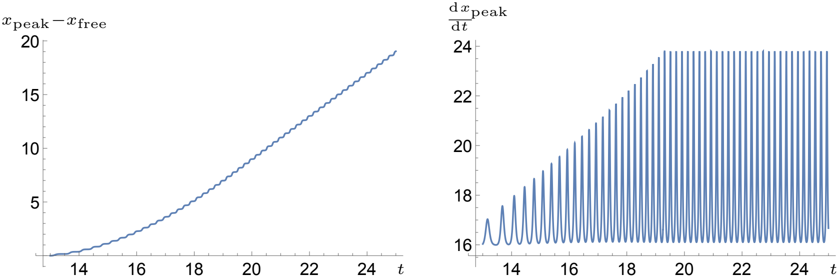


Figure: (1) Soliton peak location after subtracting the location of the free soliton. (2) Velocity of the peak soliton.
The effective velocity of the soliton is the asymptotic average of this highly oscillatory velocity profile. Here $r \equiv 1$, $\eta_1 = 0.25, \eta_2 = 1, \kappa_0 = 2$, and $\chi = 4e^{-800}$.

Theorem (GGJMcLM, '22)

The mKdV soliton + gas solution has the following asymptotic behaviour:

$$q(x,t) = q_{\mathrm{bg}}(x,t) + q_{\mathrm{sol}}(x,t) + \mathcal{O}\left(t^{-1}\right),$$

where

$$q_{\mathrm{bg}}(x,t) = (\alpha+\eta_1)\,\mathrm{dn}\left((\alpha+\eta_1)(x - 2(\eta_1^2+\alpha^2)t - x^{(\pm)}) \mid m_1\right),$$

$m_1 = \frac{4\alpha\eta_1}{(\alpha+\eta_1)^2}$, *and*

$$q_{\mathrm{sol}}(x,t) = \frac{2\left(1-\mathcal{Q}^{(\pm)}\right)^2)X^{(\pm)} + 4\mathcal{Q}^{(\pm)}Y^{(\pm)}}{\left(X^{(\pm)}\right)^2 + \left(Y^{(\pm)}\right)^2}.$$

where $\mathcal{Q}^{(\pm)}$, $X^{(\pm)}$, $Y^{(\pm)}$ *are explicit quantities in terms of* $f(\mathrm{i}\kappa_0;x,t)$, $\varphi(\mathrm{i}\kappa_0;x,t)$ *and* θ_3*-functions.*

In particular, $q_{\mathrm{sol}}(x,t) \to 0$ as $(x,t)\to\infty$ in any compact subset of $\mathcal{S}^{(\pm)}$.

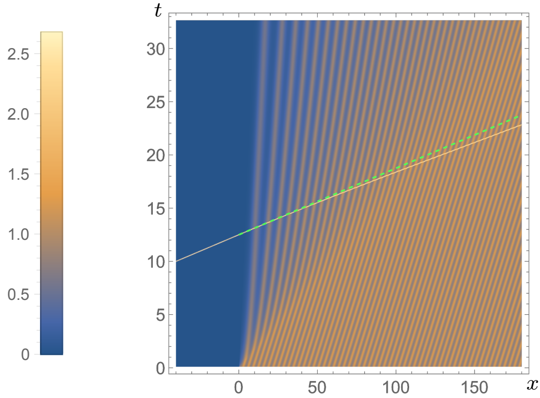


Figure: Color map plotting the leading order behaviour of a trial soliton traveling through a soliton gas. The dashed green line shows the position the trial soliton in a vacuum. Here $r \equiv 1$, $\eta_1 = 0.25$, $\eta_2 = 1$, $\kappa_0 = 2$, and $\chi = 4e^{-800}$.

Finer details...

Corollary (Phase shift)

The phase shifts $x^{(\pm)}$ of the soliton gas are different if the tracer soliton is behind ($x^{(+)}$) or in front of the wave ($x^{(-)}$) when looking at a fixed direction x/t:

$$x^{(+)} - x^{(-)} = \frac{2K\left(\frac{\eta_1^2}{\alpha^2}\right)}{\alpha}\left(1 + \int_{\mathrm{i}\alpha}^{\mathrm{i}\kappa_0} \frac{\alpha}{\mathrm{i}K\left(\frac{\eta_1^2}{\alpha^2}\right)} \frac{\mathrm{d}k}{\sqrt{(k^2-\alpha^2)(k^2-\eta_1^2)}}\right) .$$

Corollary (Giant soliton)

In the limit as $\kappa_0 \to +\infty$,

$$q_{\mathrm{sol}}(x,t) = 2\kappa_0 \operatorname{sech}\left(2\kappa_0(x - x_0 - 4\kappa_0^2 t)\right) + \mathcal{O}(1).$$

where generically the $\mathcal{O}(1)$ term is a complicated non-vanishing expression.

Velocities

Recall: dynamics is driven by the g-function

$$g(k;x,t) = \int_{\Sigma_{1,\alpha}} \ln\left(\frac{k-s}{k+s}\right) \rho(s)\mathrm{d}s$$

and the *wave phase* $\varphi(k;x,t) = g(k;x,t) + kx + 4k^3 t$.

—bf Define:

- *phase velocity*: $v_{\text{phase}}(k) = -\frac{\varphi_t}{\varphi_x}$
  - Elliptic wave velocity: $v_{\text{ell}} = v_{\text{phase}}(i\eta_1) = 2(\eta_1^2 + \alpha^2)$
  - Average soliton velocity:

$$\bar{v}_{\text{sol}}(\kappa_0) = v_{\text{phase}}(i\kappa_0) = 4\kappa_0^2 \frac{K\left(\frac{\eta_1^2}{\alpha^2}\right)}{\Pi\left(\frac{\eta_1^2}{\kappa_0^2}, \frac{\eta_1^2}{\alpha^2}\right)} + 2(\eta_1^2 + \alpha^2)$$

- *group velocity*:

$$v_{\text{group}}(k) = -\frac{\rho_t}{\rho_x} = -12\frac{k^4 + \frac{1}{2}(\eta_1^2 + \alpha^2)k^2 + c_2}{k^2 + c_0}.$$

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Soliton gas velocity

Proposition

The group velocity $v_{\text{group}} := -\frac{\rho_t}{\rho_x}$ *solves the kinetic equation (here* $k = \mathrm{i}\kappa$*)*

$$v_{\text{group}}(\kappa) = 4\kappa^2 + \frac{1}{\kappa}\int_{\eta_1}^{\alpha} \ln\left|\frac{\kappa - s}{\kappa + s}\right| \left(v_{\text{group}}(s) - v_{\text{group}}(\kappa)\right)\rho_x(\mathrm{i}s)\mathrm{d}s \ .$$

(the continuity equation is trivially satisfied)

The soliton peak

Theorem (part I: position)

For κ_0 large enough, $\exists!$ continuous, global maximum $x_{\text{peak}}(t)$ of the solution $q(x,t)$ such that:

(i) *for $t \in (0, t_1)$ (quiescent region), $x_{\text{peak}}(t) = x_0 + 4\kappa_0^2 t$;*

(ii) *for $t > (1+\epsilon)t_1$ (elliptic region),*

$$x_{\text{peak}}(t) = x^*(t) + \mathcal{O}(t^{-1})$$

where $x^*(t)$ is implicitly defined as the solution of

$$X^{(-)}(x^*(t), t) = \frac{1 - \mathcal{Q}^{(-)}(x^*(t), t)}{1 + \mathcal{Q}^{(-)}(x^*(t), t)} \frac{1 + \mathcal{Q}^{(-)}(x^*(t), t)^2}{2\kappa_0} .$$

The amplitude of the solution at the maximum is given by

$$q(x_{\text{peak}}(t), t) = q_{\text{bg}}(x^*(t), t) + 2\kappa_0 \left[1 + \frac{2\mathcal{Q}^{(-)}(x^*(t), t)}{1 + \mathcal{Q}^{(-)}(x^*(t), t)^2} \right] + \mathcal{O}(t^{-1}).$$

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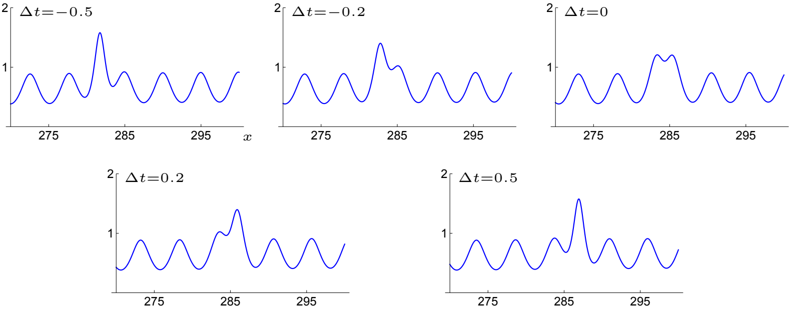


Figure: Evolution of $q(x,t)$ at times $t^* + \Delta t$ around a time $t^* = 503.17$ at which the global max becomes multivalued. Here as well $\eta_1 = 0.25, \eta_2 = 0.75, \kappa = 0.8$ and $\chi = 1.6e^{-320}$.

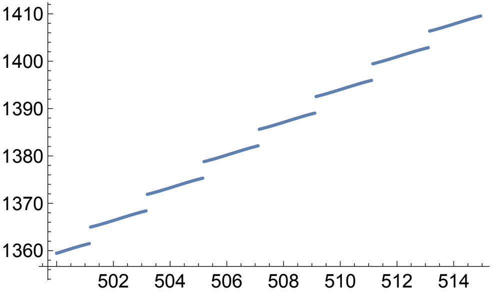


Figure: Discontinuities in the location of $x_{\text{peak}}(t)$ when the soliton eigenvalue $i\kappa_0$ is near the soliton gas spectral band $[i\eta_1, i\eta_2]$. Here $\eta_1 = 0.25, \eta_2 = 0.75, \kappa = 0.8$ and $\chi = 1.6e^{-320}$.

Trial soliton velocity

Theorem (part II: velocity)

For $t > (1+\epsilon)t_1$,

$$\dot{x}_{\text{peak}} = -\frac{2\varphi_t(\mathrm{i}\kappa_0) - \partial_t \ln \Psi(x,t;\kappa_0,\eta_1)}{2\varphi_x(\mathrm{i}\kappa_0) - \partial_x \ln \Psi(x,t;\kappa_0,\eta_1)}\Big|_{x=x_{\text{peak}}(t)} + \mathcal{O}\left(t^{-1}\right) \ .$$

Let T *be the time it takes the soliton peak* $x_{\text{peak}}(t)$ *to traverse one period of the elliptic background wave* $q_{\text{bg}}(x,t)$*. Then the* average velocity of the soliton peak over the period *satisfies*

$$\frac{1}{T}\int_t^{t+T} \dot{x}_{\text{peak}}(s)\,\mathrm{d}s = \frac{x_{\text{peak}}(t+T) - x_{\text{peak}}(t)}{T} = \bar{v}_{\text{sol}}(\kappa_0) + \mathcal{O}(t^{-1}),$$

Theorem (part III: kinetic equations)

The leading order term for the averaged soliton velocity $\bar{v}_{\rm sol}$ satisfies

$$\bar{v}_{\rm sol}(\kappa_0) = 4\kappa_0^2 + \frac{1}{\kappa_0}\int_{\eta_1}^{\alpha} \ln\left|\frac{\kappa_0 - s}{\kappa_0 + s}\right| (v_{\rm gas}(s) - \bar{v}_{\rm sol}(\kappa_0))\,\partial_x \rho(\mathrm{i}s)\,\mathrm{d}s$$

the El–Zakharov kinetic equation for the average velocity of an mKdV soliton.

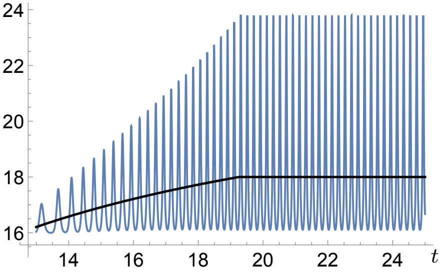


Figure: Comparison between $\bar{v}_{\rm sol}$ (black curve) and $\dot{x}_{\rm peak}$ (blue) as the soliton traverses the gas. The parameters are $r \equiv 1$, $\eta_1 = 0.25, \eta_2 = 1, \kappa_0 = 2$, and $\chi = 4e^{-800}$.

Idea of the proof

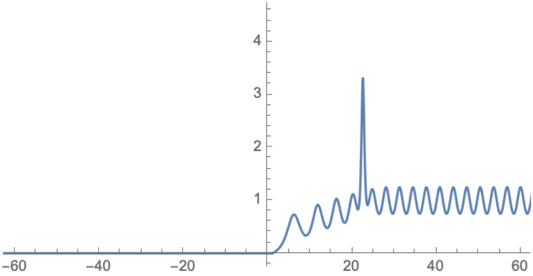
Within the elliptic region ($\frac{x}{t} > 4\eta_1^2 + \epsilon$, $\frac{x_0}{t} + 4\kappa_0^2 > 4\eta_1^2 + \epsilon$), where χ, x, t are *large* parameters.

Intuition #1: small residue conditions are negligible; large ones are negligible too.
$\Rightarrow$ the soliton peak can be isolated from identifying a space–time region where

$$\log\left|\frac{\chi}{2\kappa_0}\right| + 2\Im\varphi(\mathrm{i}\kappa_0; x, t) = \mathcal{O}(1).$$

Intuition #2: follow the characteristic lines of the phase of the background wave (make a "smart" change of coordinates $(x, t) \mapsto (s, \tau)$).

In the physical coordinates, this setting will account for the background gas q_{bg} to be *slowly varying*, while the soliton q_{sol} is passing through it.



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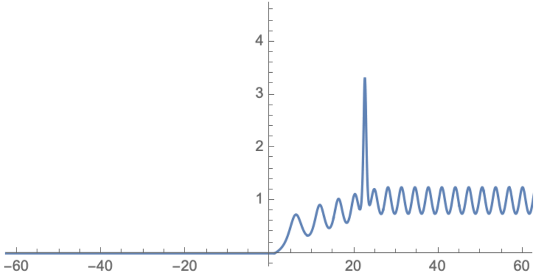

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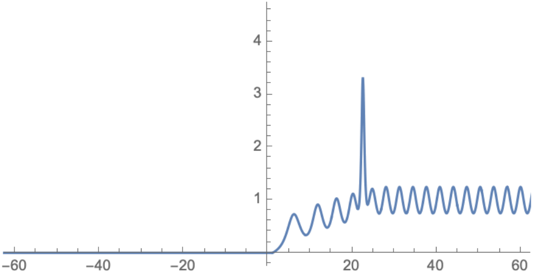
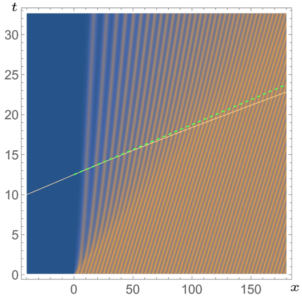
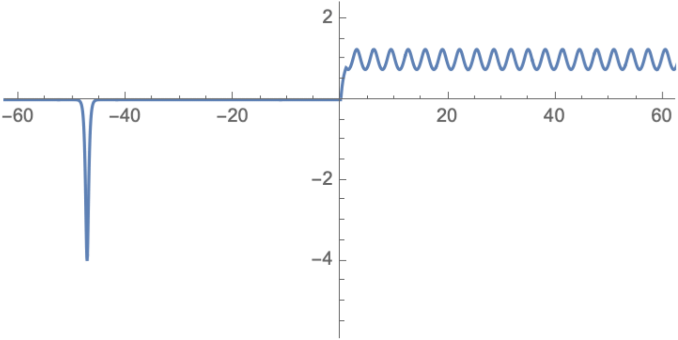


Comments

- Full description of the soliton gas in the large time regime, over the whole spatial domain.
- Rigorous description of how a particle (= soliton) interacts with a complex medium (= gas): how the medium affects the propagation of the particle (the soliton velocity) and how the particle affects the evolution of the complex medium (the phase shift in the soliton gas).
- This -regular, dense- gas is *deterministic*!
  However, the interaction dynamics is governed by the same kinetic equations of El, *et al.* (random gasses)



- For the mKDV equation other condensates are possible like a gas of breathers with interesting geometries of breathers distribution in the complex plane.
- Compare numerically, asymptotic expression obtained via Fredholm determinant with large N-soliton solutions.



Thank you!

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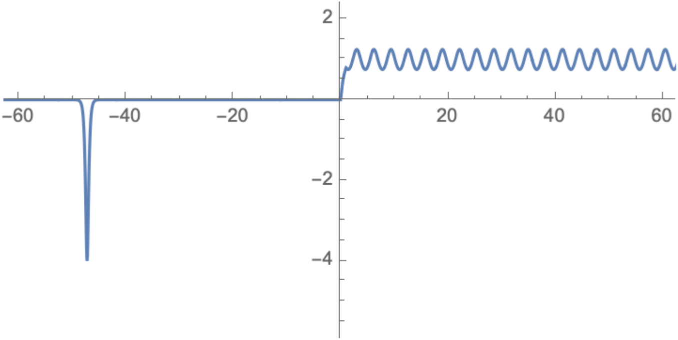

Thank you!