Analytic structure of cosmological correlators

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Crossing the bridge: New connections in number theory and physics,
Newton Institute, Cambridge, UK, August 26 2022

based on work done in collaboration with
Till Heckelbacher, Ivo Sachs and Evgeny Skvortsov,
[arXiv:2201.09626] , [arXiv:2204.07217]

Motivation

de Sitter space-time is a simple (most symmetric) model for the early (inflation)[1] and late time evolution of an accelerated expanding Universe

$$ds^2_{\rm FRW} = -dt^2 + e^{2Ht} d\vec{x}^2 \Longrightarrow ds^2_{\rm dS} = \frac{-d\eta^2 + d\vec{x}^2}{(H\eta)^2}$$

It is important to consider quantum field theory in de Sitter space for understanding the spectrum of density fluctuations in the cosmic microwave background and structure formation in the universe

If the universe were simply the motion which follows from a given scheme of equations of motion with trivial initial conditions, it could not contain the complexity we observe. Quantum mechanics provides an escape from the difficulty. (P.A.M. Dirac "The Relation between Mathematics and Physics," Proc.Roy.Soc.Edinburgh 59 (1939)

[1] [Starobinsky; Guth; Linde; Albrecht, Steinhardt; Mukahnov and Chibisov]

Motivation

Our main QFT tools are well suited for asymptotically static space-times (flat Minkowski, or AdS/CFT, for instance)

The absence of globally defined time-like Killing vector in de Sitter makes the choice of a vacuum is rather ambiguous and the definition of an asymptotic region, relevant for scattering experiments not so clear

We *analytically* evaluate correlator of conformally coupled scalars in rigid dS_4 in perturbation (higher loop Witten diagram in the bulk). We will compare with an equivalent result in (E)AdS.

We will draw from this new results for the dual CFT for conformally coupled scalar in rigid dS, and this gives explicit methods for analysis QFT in de Sitter space.

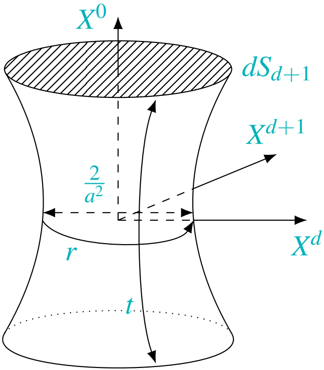
Global Observables

We are interested by global observables at the future infinity

$$\eta_{AB}X^A X^B := -(\mathbf{X}^0)^2 + \sum_{i=1}^{d+1} (\mathbf{X}^i)^2 = \frac{1}{a^2}$$

$$\partial dS_{d+1} := \left\{ \mathbf{X} \in \mathbb{RP}^{d+1}, \eta_{AB}X^A X^B = 0 \right\}$$

The boundary is $\mathbb{R}^d \cup \{\infty\}$ equipped with the Euclidean metric



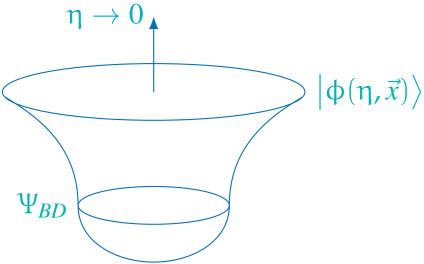
There are two global observables that one can consider in a rigid de-Sitter space-time

The wave-function

The Bunch-Davies wave-function $\Psi_{\rm BD}[\phi(\eta,\vec{x})] = \langle out|in\rangle$ is given by a path-integral over a hemisphere attached to the Poincaré patch of dS at past infinity

$$\Psi_{\rm BD}[\phi(\eta,\vec{x})] := \left\langle \phi(\eta,\vec{x}) \left| U(\eta,-\infty) \right| 0_{BD} \right\rangle$$

$$= \lim_{\eta'\to-\infty(1+i\varepsilon)} \int_{\substack{\phi(0,x)=\phi_0(x)\\ \phi(\eta',x)=0}} \mathcal{D}\phi e^{iS[\phi]}$$



From which we can compute correlators in an in-out formalism

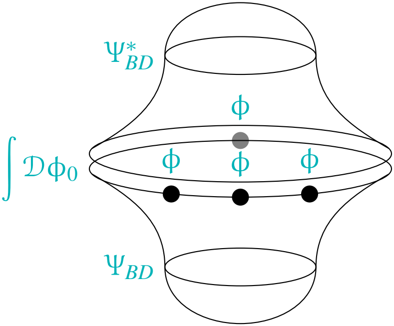
$$\frac{\delta}{\delta\phi_0}\cdots\frac{\delta}{\delta\phi_0}\Psi_{\rm DB}[\phi_0] = \langle \mathcal{O}\cdots\mathcal{O}\rangle$$

The cosmological correlators

In this work we will only consider the cosmological correlators computed in the Schwinger-Keldysh (in-in) formalism with respect to the Bunch-Davis vacuum

$$\left\langle \prod_{i=1}^{m} \phi(\vec{x}_i) \right\rangle =$$

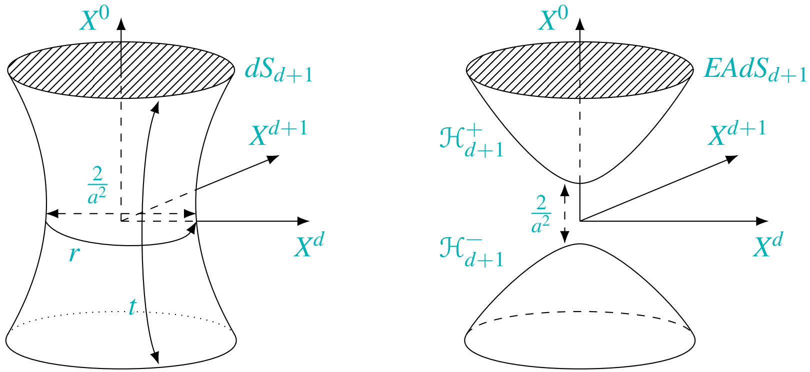
$$\int \mathcal{D}\phi_0 \Psi^*_{BD}[\phi_0]\Psi_{BD}[\phi_0] \prod_{i=1}^{m} \phi(\vec{x}_i)$$



Global dS and EAdS geometry

(Global) de Sitter and (Euclidean) anti-de Sitter space are both embedded in 5-dimensional hyperboloid with -1 for EAdS, $+1$ for dS

$$-(\mathbf{X}^0)^2 + \sum_{i=1}^{3} (\mathbf{X}^i)^2 + (\mathbf{X}^4)^2 = \frac{\pm 1}{a^2}$$



dS and EAdS Poincaré Patch

de Sitter Poincaré patch

$$\mathbf{X}^0 = \frac{1}{\sqrt{2}az}\left(1+\frac{\vec{x}^2}{2}-\frac{\eta^2}{2}\right),$$

$$\mathbf{X}^i = \frac{x^i}{a\eta},$$

$$X^{d+1} = \frac{1}{\sqrt{2}a\eta}(1-\frac{\vec{x}^2}{2}+\frac{\eta^2}{2})$$

$$\mathcal{H}_4^- := \left\{X := (\vec{x},\eta),\, \vec{x}\in\mathbb{R}^3, \eta<0\right\}$$

(E)AdS Poincré patch

$$\mathbf{X}^0 = \frac{1}{\sqrt{2}az}\left(1-\frac{\vec{x}^2}{2}-\frac{z^2}{2}\right),$$

$$\mathbf{X}^i = \frac{x^i}{az},$$

$$X^{d+1} = \frac{1}{\sqrt{2}az}(1+\frac{\vec{x}^2}{2}+\frac{z^2}{2})$$

$$\mathcal{H}_4^+ := \left\{X := (\vec{x},z),\, \vec{x}\in\mathbb{R}^3, z>0\right\}$$

$$(z,a)\leftrightarrow(i\eta,-ia),\qquad ds^2 = \frac{dz^2+d\vec{x}^2}{a^2z^2}$$

Auxiliary action for Euclidean AdS action

The closed time evolution between two in-states (the Schwinger-Keldysh formalism) from the infinite past can be expressed by a path integral with closed time curves.

Then a correlation function is given by taking functional derivatives of the time and anti-time ordered sources j_T and j_A of the partition function

$$Z[j_T, j_A] = \int \mathcal{D}\phi_T \mathcal{D}\phi_A e^{iS_c + i\int(\phi_T j_T + \phi_A j_A)},$$

Under the Wick rotation $\eta = z e^{\pm i\frac{\pi}{2}}$ the classical solutions of a free scalar field in de Sitter reads

$$\phi(z e^{\pm i\frac{\pi}{2}}, \vec{x}) = e^{\pm i\frac{\pi}{2}\Delta_+}\phi^+(z, \vec{x}) + e^{\pm i\frac{\pi}{2}\Delta_-}\phi^-(z, \vec{x}),$$

and $\phi^\pm(z, \vec{x}) \to z^{\Delta_\pm}$ for $z \to 0$ on the Poincaré patch $\Delta_+ + \Delta_- = d$

Auxiliary action for Euclidean AdS action

For a potential $V(\phi) = \frac{\lambda}{4!}\phi^4$ and non-minimally coupled conformally coupled scalar with $\Delta_+ = \frac{d+1}{2}$ and $\Delta_- = \frac{d-1}{2}$, with effective mass $m^2 = -a^2\Delta(\Delta - d)$ with odd boundary dimensions $d = 3$ the action becomes[2]

$$iS_c = -\int_0^\infty \frac{dz d^3\vec{x}}{z^4}\Big[-\Big((\partial\phi^+)^2 - m^2\phi^{+2}\Big) + \Big((\partial\phi^-)^2 - m^2\phi^{-2}\Big) - \frac{2\lambda}{4!}\Big(\phi^{+4} - 6\phi^{+2}\phi^{-2} + \phi^{-4}\Big)\Big].$$

The kinetic term in the action is not necessarily positive, leading to ghost-like behaviour of one of the fields. This would be a problem if we wanted to interpret this action as describing a bulk theory in EAdS, however, we only us this action as a book keeping device to keep track of the correct relative prefactors in the expansion.

2 [L. di Pietro, V. Gorbenko, S. Komatsu, [arXiv:2108.01695]]

EAdS propagator

The bulk-to-bulk propagator $\Lambda(\mathbf{X},\mathbf{Y};\Delta)$ between two bulk points in EAdS is a solution of

$$\left(-\Box + a^2\Delta(d-\Delta)\right)\Lambda(\mathbf{X},\mathbf{Y}) = \frac{1}{\sqrt{|g|}}\delta^4(\mathbf{X}-\mathbf{Y}),$$

To fix the normalization of the Dirichlet and Neumann Green function we demand that in the flat space limit, $a\to 0$, their singularity agrees with the flat space Green function. The properly normalized time-ordered propagator is

$$\begin{aligned}\Lambda(\mathbf{X},\mathbf{Y};\Delta_\pm) := \left(\frac{a}{2\pi}\right)^2 \frac{\Gamma\left(\frac{\Delta_\pm}{2}\right)\Gamma\left(\frac{\Delta_\pm+1}{2}\right)}{\Gamma\left(\Delta_\pm-\frac{1}{2}\right)} K(\mathbf{X},\mathbf{Y})^{\Delta_\pm} \\ \times\, {}_2F_1\left(\frac{\Delta_\pm}{2},\frac{\Delta_\pm+1}{2};\Delta_\pm-\frac{1}{2};K(\mathbf{X},\mathbf{Y})^2+i\varepsilon\right)\end{aligned}$$

$$K(\mathbf{X},\mathbf{Y}) = -\frac{1}{a^2\mathbf{X}\cdot\mathbf{Y}} = \frac{2zw}{(\vec{x}-\vec{y})^2+z^2+w^2}$$

EAdS propagator as flat space propagators

For $\Delta = 1$ and $\Delta = 2$ the bulk-to-bulk propagator is then expressed in terms of the conformal massless flat space propagator as[3]

$$\Lambda(\mathbf{X},\mathbf{Y};1) = -\left(\frac{a}{2\pi}\right)^2 \left(G(X,Y) - G(X,\sigma(Y))\right),$$
$$\Lambda(\mathbf{X},\mathbf{Y};2) = -\left(\frac{a}{2\pi}\right)^2 \left(G(X,Y) + G(X,\sigma(Y))\right).$$

$$G(X,Y) := \frac{u\cdot X\, u\cdot Y}{\|X-Y\|^2 + i\varepsilon}, \quad \|X\|^2 := \vec{x}^2 + z^2, \quad u := (\vec{0},1),$$

The anti-podal map will play an important role for rewritting the integral over the full covering space $\sigma(X) := \sigma(\vec{x},z) := (\vec{x},-z)$

There is a natural $+i\varepsilon$ prescription corresponding to the prescription on the general bulk-to-bulk propagator in AdS[4]

3 [T. Heckelbacher, I. Sachs, E. Skvortsov, P. Vanhove, [arXiv:2201.09626]]

4 [S. Avis et al. "Quantum Field Theory in Anti-De Sitter Space-Time", Phys.Rev.D 18 (1978) 3565]

Four-point functions

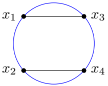
The EAdS action couples the two fields ϕ^- of dimension $\Delta_- = 1$ and ϕ^+ with dimension $\Delta_+ = 2$

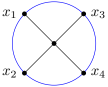
$$\left\langle \phi_0(\vec{x_1})\phi_0(\vec{x_2})\phi_0(\vec{x_3})\phi_0(\vec{x_4})\right\rangle = \eta_0^{4\Delta_-} \left\langle \phi^-(\vec{x_1})\phi^-(\vec{x_2})\phi^-(\vec{x_3})\phi^-(\vec{x_4})\right\rangle$$
$$+ \eta_0^{2(\Delta_-+\Delta_+)} \Big(\left\langle \phi^+(\vec{x_1})\phi^+(\vec{x_2})\phi^-(\vec{x_3})\phi^-(\vec{x_4})\right\rangle$$
$$+ \left\langle \phi^+(\vec{x_1})\phi^-(\vec{x_2})\phi^+(\vec{x_3})\phi^-(\vec{x_4})\right\rangle$$
$$+ \left\langle \phi^+(\vec{x_1})\phi^-(\vec{x_2})\phi^-(\vec{x_3})\phi^+(\vec{x_4})\right\rangle \Big) + \eta_0^{4\Delta_+} \left\langle \phi^+(\vec{x_1})\phi^+(\vec{x_2})\phi^+(\vec{x_3})\phi^+(\vec{x_4})\right\rangle .$$

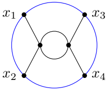
The leading contributions in the late time expansions are given by calculating the EAdS correlators of the field ϕ^-. Note however, that this field alone will not give a consistent CFT at the boundary, since there will be mixing interaction vertices between ϕ^- and ϕ^+.

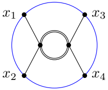
To be able to describe the CFT on the boundary we have to take into account the subleading terms in the late time expansion of the cosmological correlator as well.

Four-point functions

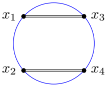
$\langle \phi^-(x_1)\phi^-(x_2)\phi^-(x_3)\phi^-(x_4)\rangle = \Bigg($ 

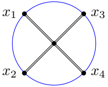
$+\,2$ perm. $\Bigg) - \lambda$ 

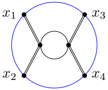
$+\,\frac{\lambda^2}{2}\Bigg($ 

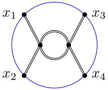
$+\,2$ perm. $\Bigg) + \frac{\lambda^2}{2}\Bigg($ 

$+\,2$ perm. $\Bigg) + \mathcal{O}(\lambda^3).$

$\langle \phi^+(x_1)\phi^+(x_2)\phi^+(x_3)\phi^+(x_4)\rangle = \Bigg($ 

$+\,2$ perm. $\Bigg) - \lambda$ 

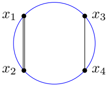
$+\,\frac{\lambda^2}{2}\Bigg($ 

$+\,2$ perm. $\Bigg) + \frac{\lambda^2}{2}\Bigg($ 

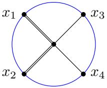
$+\,2$ perm. $\Bigg) + \mathcal{O}(\lambda^3).$

Four-point functions

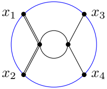
$$\langle \phi^+(x_1)\phi^+(x_2)\phi^-(x_3)\phi^-(x_4)\rangle =$$



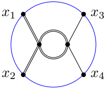
$$+\lambda$$



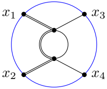
$$-\lambda^2\Bigg(\frac{1}{2}$$



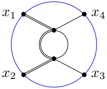
$$+\frac{1}{2}$$



$$+$$



$$+$$



$$\Bigg) + \mathcal{O}(\lambda^3).$$

Tadpoles and self-energy corrections are absorbed into the conformal dimension of the boundary operator

$$\langle \phi(x_1)\phi(x_2)\rangle = \; x_1$$

$$x_2 \; + \frac{\lambda}{2} \; x_1$$

$$x_2 \; + \mathcal{O}(\lambda^2)$$

$$\Lambda(K;\Delta) = \frac{a^2K^\Delta}{4\pi^2(1-K^2)}, \quad m^2 = a^2\Delta(d-\Delta), \quad \Delta = \frac{3\pm 1}{2} + \sum_{L\geqslant 1}\gamma_L\lambda^L$$

The disconnected parts

The disconnected part is given by the product of two-point functions

$$\left\langle \phi^{\Delta\pm}(x_1)\phi^{\Delta\pm}(x_2)\right\rangle\left\langle \phi^{\Delta\pm}(x_3)\phi^{\Delta\pm}(x_4)\right\rangle+\left\langle \phi^{\Delta\pm}(x_1)\phi^{\Delta\pm}(x_3)\right\rangle\left\langle \phi^{\Delta\pm}(x_2)\phi^{\Delta\pm}(x_4)\right\rangle$$
$$+\left\langle \phi^{\Delta\pm}(x_1)\phi^{\Delta\pm}(x_4)\right\rangle\left\langle \phi^{\Delta\pm}(x_2)\phi^{\Delta\pm}(x_3)\right\rangle=\frac{2^2a^4}{(4\pi^2)^2}\frac{1}{x_{12}^{2\Delta\pm}x_{34}^{2\Delta\pm}}\left(1+v^{\Delta\pm}+\frac{v^{\Delta\pm}}{(1-Y)^{\Delta\pm}}\right)$$

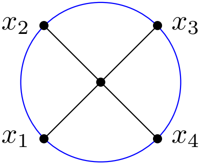
$$\left\langle \phi^{+}(x_1)\phi^{+}(x_2)\right\rangle\left\langle \phi^{-}(x_3)\phi^{-}(x_4)\right\rangle=\frac{2^2a^4}{(4\pi^2)^2}\frac{1}{x_{12}^4x_{34}^2}\,.$$

with the cross-ratio of the distance on the boundary $x_{ij}^2=|\vec{x_i}-\vec{x_j}|^2$

$$v=\frac{x_{12}^2x_{34}^2}{x_{14}^2x_{23}^2}=\zeta\bar\zeta;\qquad 1-Y=\frac{x_{13}^2x_{24}^2}{x_{14}^2x_{23}^2}=(1-\zeta)(1-\bar\zeta)\,.$$

Cross diagram

The first order perturbation in $\lambda\phi^4$ theory is the cross Witten diagram $\mathcal{W}_0^{\Delta,D}$



$$= \frac{1}{2}\frac{v^\Delta}{(x_{12}^2 x_{34}^2)^\Delta}\int_{\mathbb{R}^D}\frac{d^DX}{(u\cdot X)^D}\left(\frac{(u\cdot X)^4}{\|X\|^2\|X-u_\zeta\|^2\|X-u_1\|^2}\right)^\Delta$$

with $u_1^2=1$, $u_\zeta^2=\zeta\bar\zeta$, $u_1\cdot u_\zeta=(1-\zeta)(1-\bar\zeta)$ and $u\cdot u_1=u\cdot u_\zeta=0$ The parametric presentation

$$\mathcal{W}_0^{\Delta,D}=\frac{1}{2}\frac{i^{4\Delta-D}\pi^{\frac{D+1}{2}}}{\Gamma\left(\frac{D+1}{2}-2\Delta\right)\Gamma(\Delta)^2}\frac{v^\Delta}{(x_{12}^2x_{34}^2)^\Delta}I_\times^\Delta(\zeta,\bar\zeta)$$

This is the D function of [F. A. Dolan and H. Osborn, [hep-th/0011040]]

Cross diagram

With the (space-time) dimension independent integral

$$I^{\Delta}_{\times}(\zeta,\bar\zeta)=\int\limits_{\alpha_i\geqslant 0}\frac{\prod_{i=1}^{3}d\alpha_i\alpha_i^{\Delta-1}}{(\alpha_1+\alpha_2+\alpha_3)^{\Delta}(\alpha_1\alpha_2+\alpha_1\alpha_3\zeta\bar\zeta+\alpha_2\alpha_3(1-\zeta)(1-\bar\zeta))^{\Delta}}$$

It is immediate to show that this integral statisfies the recursion relation

$$\sum_{n=0}^{4}c(\Delta,n)I^{\Delta+n}_{\times}(\zeta,\bar\zeta)=0$$

which implies that for all $\Delta\geqslant 1$ integer we have combination of single-valued polylogarithm of weight at most 2

$$\begin{aligned}I^{\Delta}_{\times}(\zeta,\bar\zeta)=&\frac{c_1^{\Delta}(\zeta,\bar\zeta)}{(\zeta-\bar\zeta)^{4(\Delta-1)}}\frac{4iD(\zeta,\bar\zeta)}{\zeta-\bar\zeta}+\frac{c_2^{\Delta}(\zeta,\bar\zeta)}{(\zeta-\bar\zeta)^{4(\Delta-1)}}\log(\zeta\bar\zeta)\\&+\frac{c_3^{\Delta}(\zeta,\bar\zeta)}{(\zeta-\bar\zeta)^{4(\Delta-1)}}\log((1-\zeta)(1-\bar\zeta))+\frac{c_4^{\Delta}(\zeta,\bar\zeta)1}{(\zeta-\bar\zeta)^{4(\Delta-1)}}\end{aligned}$$

Cross diagram

$$I^{\Delta}_{\times}(\zeta,\bar\zeta) = \frac{c_1^{\Delta}(\zeta,\bar\zeta)}{(\zeta-\bar\zeta)^{4(\Delta-1)}}\frac{4iD(\zeta,\bar\zeta)}{\zeta-\bar\zeta} + \frac{c_2^{\Delta}(\zeta,\bar\zeta)}{(\zeta-\bar\zeta)^{4(\Delta-1)}}\log(\zeta\bar\zeta)$$
$$+ \frac{c_3^{\Delta}(\zeta,\bar\zeta)}{(\zeta-\bar\zeta)^{4(\Delta-1)}}\log((1-\zeta)(1-\bar\zeta)) + \frac{c_4^{\Delta}(\zeta,\bar\zeta)1}{(\zeta-\bar\zeta)^{4(\Delta-1)}}$$

where $c_r^{\Delta}(\zeta,\bar\zeta)$ are polynomial in ζ and $\bar\zeta$, and with $D(\zeta,\bar\zeta)$ is the Bloch-Wigner dilogarithm

$$D(\zeta,\bar\zeta) = \frac{1}{2i}\left(\mathrm{Li}_2\left(\zeta\right) - \mathrm{Li}_2\left(\bar\zeta\right) - \frac{1}{2}\log(\zeta\bar\zeta)\left(\mathrm{Li}_1\left(\zeta\right) - \mathrm{Li}_1\left(\bar\zeta\right)\right)\right)$$

Despite the apparent singularity for $\bar\zeta = \zeta$ the expression is regular on the real slice and the expression is single-valued on $\mathbb{C}\backslash\{0,1\}$

Cross diagram

For $\Delta = 1$

$$\mathcal{W}_0^{1,4}(\zeta,\bar\zeta) = \frac{\pi^2}{x_{12}^2 x_{34}^2}\zeta\bar\zeta\frac{2iD(\zeta,\bar\zeta)}{\zeta-\bar\zeta}$$

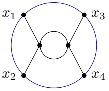
For $\Delta = 2$

$$\begin{aligned}\mathcal{W}_0^{2,4}(\zeta,\bar\zeta) &= \frac{3\pi^2(\zeta\bar\zeta)^2}{4x_{12}^4 x_{34}^4}\\ &\times\Big(\frac{4\zeta^2\bar\zeta^2-(\zeta+\bar\zeta)^3+2\zeta\bar\zeta(\zeta+\bar\zeta)^2+2(\zeta+\bar\zeta)^2-8\zeta\bar\zeta(\zeta+\bar\zeta)+4\zeta\bar\zeta}{(\zeta-\bar\zeta)^4}\frac{2iD(\zeta,\bar\zeta)}{\zeta-\bar\zeta}\\ &\qquad+\frac{(\zeta+\bar\zeta)^2-3\zeta\bar\zeta(\zeta+\bar\zeta)+2\zeta\bar\zeta}{(\zeta-\bar\zeta)^4}\log(\zeta\bar\zeta)\\ &\qquad+\frac{3\zeta\bar\zeta(\zeta+\bar\zeta)-2(\zeta+\bar\zeta)^2+3(\zeta+\bar\zeta)-4\zeta\bar\zeta}{(\zeta-\bar\zeta)^4}\log((1-\zeta)(1-\bar\zeta))+\frac{1}{(\zeta-\bar\zeta)^2}\Big)\end{aligned}$$

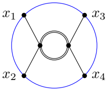
One-loop graphs in de Sitter

$$\langle \phi^-(x_1)\phi^-(x_2)\phi^-(x_3)\phi^-(x_4)\rangle|_{\text{one-loop}} =$$

$$\frac{\lambda^2}{2}\Bigg($$



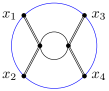
$$+\,2\text{ perm.}\Bigg) + \frac{\lambda^2}{2}\Bigg($$



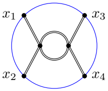
$$+\,2\text{ perm.}\Bigg)$$

$$\langle \phi^+(x_1)\phi^+(x_2)\phi^+(x_3)\phi^+(x_4)\rangle|_{\text{one-loop}} =$$

$$\frac{\lambda^2}{2}\Bigg($$



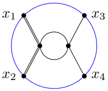
$$+\,2\text{ perm.}\Bigg) + \frac{\lambda^2}{2}\Bigg($$



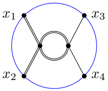
$$+\,2\text{ perm.}\Bigg)$$

$$\langle \phi^+(x_1)\phi^+(x_2)\phi^-(x_3)\phi^-(x_4)\rangle_{\text{one-loop}} =$$

$$-\lambda^2\Bigg(\frac{1}{2}$$



$$+\,\frac{1}{2}$$



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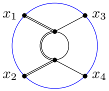

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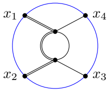


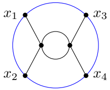
$$\Bigg)$$

One-loop graphs in de Sitter

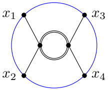
By regrouping contributions from the $\Delta = 1$ and $\Delta = 2$ fields propagating in the loops one obtain some simplifications

$$\Lambda(X_1, X_2; 1)^2 + \Lambda(X_1, X_2; 2)^2 = 2\frac{(u\cdot X_1)^2\,(u\cdot X_2)^2}{\|X_1 - X_2\|^4} + 2\frac{(u\cdot X_1)^2\,(u\cdot \sigma(X_2))^2}{\left\|X_1 - \sigma(X_2)\right\|^4}$$

After unfolding the integral to the whole space $\mathbb{R}^4$



$+$



$$\propto \int_{(\mathcal{H}_D^+)^2} \frac{d^DX d^DY}{(u\cdot X)^4 (u\cdot Y)^4} \frac{(u\cdot X)^2 (u\cdot Y)^2}{\prod_{i=1}^2 \|X - x_i\|^2 \prod_{j=3}^4 \left\|Y - x_j\right\|^2}$$

$$\times \left(\frac{(u\cdot X_1)^2\,(u\cdot X_2)^2}{\|X_1 - X_2\|^4} + \frac{(u\cdot X_1)^2\,(u\cdot \sigma(X_2))^2}{\left\|X_1 - \sigma(X_2)\right\|^4}\right)$$

$$\propto \int_{(\mathbb{R}^D)^2} \frac{d^DX d^DY}{\|X - Y\|^4 \|X - x_1\|^2 \|X - x_2\|^2 \|Y - x_3\|^2 \|Y - x_4\|^2}$$

Dimensionally regulated Witten diagrams

We regulate the coincident bulk points divergences by using a dimensional regularisation

$$\int \prod_{i=1}^{L} \frac{d^{4-2\epsilon} X_i \, \mu^{2\epsilon}}{(u \cdot X_i)^4}$$

Changing the dimension as $\int \prod_{i=1}^{L} \frac{d^{4-2\epsilon} X_i}{(u \cdot X_i)^{4-2\epsilon}}$ does **not** regulate the integration.

This regularisation breaks the EAdS-invariance at intermediate steps but the symmetry is recovered by consider the higher terms in the $D - 4 = -2\epsilon$

Because the regularisation breaks conformal invariance we get ϵ in the propagators

$$\mathcal{W}_{1,\text{div}}^{\Delta,4-2\epsilon,s} = \frac{(\zeta\bar{\zeta})^{\Delta}}{(x_{12}^2 x_{34}^2)^{\Delta}} \int_{\mathbb{R}^{2D}} \frac{\mathrm{d}^{4-2\epsilon} X_1 \mathrm{d}^{4-2\epsilon} X_2 (u \cdot X_1)^{2\Delta-2} (u \cdot X_2)^{2\Delta-2}}{\|X_1\|^{2\Delta} \|X_1 - u_\zeta\|^{2\Delta} \|X_2 - u_1\|^{2\Delta-4\epsilon} \|X_1 - u_1\|^{-4\epsilon} \|X_1 - X_2\|^4}$$

One-loop graphs in de Sitter

Summing over all the channels the one-loop contribution reads

$$\left\langle \phi^-(x_1)\phi^-(x_2)\phi^-(x_3)\phi^-(x_4)\right\rangle|_{\text{one-loop}} =$$

$$\frac{2^2 a^4}{(4\pi^2)^2}\,\frac{2^2\lambda^2}{(4\pi^2)^4}\left(-\frac{3\pi^2}{\epsilon}\mathcal{W}_0^{1111,4-4\epsilon}(v,Y)+\frac{\pi^4 v}{2x_{12}^2x_{34}^2}\sum_{i\in\{s,t,u\}}L_0^{1,i}(v,Y)\right)$$

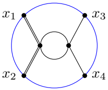
with a similar result for the one-loop contribution to the correlator $\left\langle \phi^+(x_1)\phi^+(x_2)\phi^+(x_3)\phi^+(x_4)\right\rangle|_{\text{one-loop}}$

The finite pieces $L_0^{\Delta,i}(v,Y)$ are single-valued up to weight 3 Mpl in ζ and $\bar\zeta$

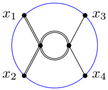
One-loop graphs in de Sitter

For the mixed correlator

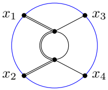
$$\langle \phi^+(x_1)\phi^+(x_2)\phi^-(x_3)\phi^-(x_4)\rangle|_{\text{one-loop}} = -\lambda^2 \Bigg(\frac{1}{2}$$



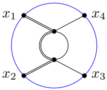
$$+\ \frac{1}{2}$$



$$+$$

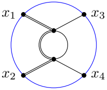


$$+$$



$$\Bigg)$$

We can again unfold the region of integration $(\mathcal{H}_D^+)^2$ to $\mathbb{R}^{2D}$

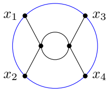


$$\propto \int_{(\mathbb{R}^D)^2} \frac{d^D X d^D Y (u\cdot X)(u\cdot Y)}{\|X-Y\|^4 \|X-x_1\|^4 \|Y-x_2\|^4 \|X-x_3\|^2 \|Y-x_4\|^2}$$

The one-loop four-point contribution has a divergent piece proportional to the cross and a finite piece given by single-valued MPL of weight at most 3.

One-loop in EAdS

In EAdS we do not have the mixing between the field and its shadow, so for $\Delta = 1$ we have



$$= \frac{2^{4\Delta} a^4}{(4\pi^2)^6} \int_{(\mathbb{R}^D)^2} \frac{d^D X d^D Y (u\cdot X)^{2\Delta-2} (u\cdot Y)^{2\Delta-2}}{\|X-Y\|^4 \|X-x_1\|^2 \|X-x_2\|^2 \|Y-x_3\|^2 \|Y-x_4\|^2}$$

$$\propto -\frac{3\pi^2}{\epsilon} \mathcal{W}_0^{1,4-4\epsilon}(v,Y) + \frac{\pi^4 v}{2x_{12}^2 x_{34}^2} \sum_{i\in\{s,t,u\}} L_0^{1,i}(v,Y) + \frac{\pi^4 v}{x_{12}^2 x_{34}^2} \sum_{i\in\{s,t,u\}} L_0'^{i}(v,Y)$$

where L_0' is an elliptic integral[5]

$$L_0'(x,y,z) = \int_1^\infty d\lambda \int_0^\infty ds \int_0^1 dr \frac{\log(1+\lambda s)}{4\lambda\sqrt{(1+s)(1+\lambda s)}(sr(1-r)x + ry + (1-r)z)}$$

The dS computation is *simpler* than the (E)AdS computation because of cancellations between the fields and its shadow

5 [I. Bertan, I. Sachs, E. D. Skvortsov, [arXiv:1810.00907]], [T. Heckelbacher, I. Sachs, E. Skvortsov, P. Vanhove, [arXiv:2201.09626]]

Renormalization

The (UV) divergences from colliding bulk points, have been regulated in dimensional regularisation which has broken the conformal symmetry

$$\int \prod_{i=1}^{L} \frac{d^{4-2\epsilon X_i} \mu^{2\epsilon}}{z^4}$$

The EAdS-invariance is recovered by consider the higher terms in the $D-4=-2\epsilon$

$$\begin{aligned}\mathcal{W}_0^{1,4-4\epsilon}(\zeta,\bar\zeta) &= \frac{1}{2}\frac{\zeta\bar\zeta}{(x_{12}x_{34})^2}\int_{\mathbb{R}^4} \frac{\mathrm{d}^{4-4\epsilon}X}{\|X\|^2\|X-u_1\|^{2(1-4\epsilon)}\|X-u_\zeta\|^2}\\ &= \mathcal{W}_0^{1,4}(\zeta,\bar\zeta) + \epsilon\mathcal{W}_{0,\epsilon}^{1,4}(\zeta,\bar\zeta) + O(\epsilon^2)\end{aligned}$$

The ϵ order term has order 3 (single-valued) MPL, and so on at higher orders

Renormalization

Recall that have a ϕ^4 theory

$$\mathcal{S} = \int \sqrt{g}\left(\frac{1}{2}(\partial\phi)^2 + m^2\phi^2 + \frac{\lambda}{4!}\phi^4\right) d^4x$$

We renormalise as usual $\lambda = \lambda_R\,(a\mu)\mu^{2\epsilon} + \delta\lambda$. The renormalised coupling λ_R is finite and dimensionless in any dimension due to the factor $\mu^{2\epsilon}$ where μ has the dimension of length

$$\frac{2^{\sum_i \Delta_i} a^4}{(8\pi^2)^4}(\mu a)^{4\epsilon}\left(2\lambda_R \mathcal{W}_0^{\Delta_1\Delta_2\Delta_3\Delta_4,4-4\epsilon} + \frac{\lambda_R^2}{16\pi^4}\frac{3\pi^2}{\epsilon}\mathcal{W}_0^{\Delta_1\Delta_2\Delta_3\Delta_4,4-4\epsilon}\right)$$
$$=: \frac{2^{\Delta_1+\cdots+\Delta_4}a^4 2}{(8\pi^2)^4}\mu^{2\epsilon}\lambda\mathcal{W}_0^{\Delta_1\Delta_2\Delta_3\Delta_4,4-4\epsilon}.$$

This determines the counter-term and the β-function coincides with the flat space result for the ϕ^4 theory

$$\delta\lambda = -\frac{3\lambda_R^2\mu^{2\epsilon}}{32\pi^2\epsilon}, \qquad \beta = \frac{3\lambda_R^2}{16\pi^2} + \mathcal{O}(\lambda_R^3)$$

Renormalization

$$\left\langle \phi^{\pm}(x_1)\phi^{\pm}(x_2)\phi^{\pm}(x_3)\phi^{\pm}(x_4)\right\rangle = \frac{2^{2\Delta_\pm}a^4}{(8\pi^2)^2}\left[\frac{1}{x_{12}^4x_{34}^4}\left(1+v^{\Delta_\pm}+\frac{v^{\Delta_\pm}}{(1-Y)^{\Delta_\pm}}\right)\right.$$
$$\left. -\frac{2^{2\Delta_\pm 2}\lambda_R}{(8\pi^2)^2}\mathcal{W}_0^{\Delta_\pm\Delta_\pm\Delta_\pm\Delta_\pm,4}+\frac{2^{2\Delta_\pm 4}\lambda_R^2}{(8\pi^2)^4}\sum_{i\in\{s,t,u\}}\mathcal{W}_{1,\mathrm{finite}}^{\Delta_\pm\Delta_\pm\Delta_\pm\Delta_\pm,i}\right]$$

$$\left\langle \phi^{+}(x_1)\phi^{+}(x_2)\phi^{-}(x_3)\phi^{-}(x_4)\right\rangle = \frac{a^4}{8\pi^4}\left[\frac{1}{x_{12}^4x_{34}^2}+\frac{\lambda_R}{4\pi^4}\mathcal{W}_0^{2211,4}+\frac{\lambda_R^2}{128\pi^8}\sum_{i\in\{s,t,u\}}\mathcal{W}_{1,\mathrm{finite}}^{2211,i}\right]$$

The renormalized four-point function transforms homogeneously under dilatations on the boundary, **we have restored the conformal symmetry on the boundary**, which was broken by the dimensional regularisation at intermediate steps

One can use as well some cutoff regularisation that preserves the (A)dS symmetry.[6] The physical observable are independent of the renormalisation scheme[7]

6 [Bertan, Sachs]

7 [T. Heckelbacher, I. Sachs, E. Skvortsov, P. Vanhove [arXiv:2201.09626]]

Holography in dS and AdS

Similar to AdS, one can define a conformal boundary for dS given by a space-like surface at future infinity. But it not possible to fix boundary conditions in the same way as in AdS since this is incompatible with unitary time evolution.

Nevertheless, one expects a CFT description of the bulk theory in dS on the boundary since the symmetry group of dS acts on the future boundary as the euclidean conformal group.

We find that the cosmological correlators obey several CFT consistency conditions at different loop orders reflecting the fact the boundary theory is in fact a CFT.

Holography in dS

We perform the conformal block expansion[8] on the generalised free fields double trace operators with dimension $\Delta_1 + \Delta_2 + 2n + l$

$$\mathcal{O}_{\Delta_1}\mathcal{O}_{\Delta_2} = 1 + \sum_{n,l} A_{n,l}^{\frac{1}{2}}[\mathcal{O}_{\Delta_1}, \mathcal{O}_{\Delta_2}]_{n,l}, \quad [\mathcal{O}_{\Delta_1}, \mathcal{O}_{\Delta_2}]_{n,l} \sim \mathcal{O}_{\Delta_1} \overleftrightarrow{\partial}^{2n} \overleftrightarrow{\partial}_{\mu_1} \cdots \overleftrightarrow{\partial}_{\mu_n} \mathcal{O}_{\Delta_2}$$

We have mixing between the double trace operators in the double OPE and the two-point function between the operators $[\mathcal{O}_1\mathcal{O}_1]_{n+1,l}$ and $[\mathcal{O}_2\mathcal{O}_2]_{n,l}$ does not vanish. So for performing the conformal block expansion we choose a basis of operators $\mathcal{O}^S_{n,l}$ and $\mathcal{O}^A_{n,l}$ both with scaling dimension $\Delta^{S/A}_{n,l} = 2 + 2n + l + \mathcal{O}(\lambda)$ and spin l such that they are orthogonal, i.e. at $\mathcal{O}(\lambda^0)$ they have the two point functions

$$\left\langle \mathcal{O}^S_{n,l}(x_1)\mathcal{O}^A_{n,l}(x_2) \right\rangle = 0;$$

$$\left\langle \mathcal{O}^S_{n,l}(x_1)\mathcal{O}^S_{n,l}(x_2) \right\rangle = \left\langle \mathcal{O}^A_{n,l}(x_1)\mathcal{O}^A_{n,l}(x_2) \right\rangle = \frac{1}{2}\left\langle [\mathcal{O}_1\mathcal{O}_1]_{n,l}(x_1)[\mathcal{O}_1\mathcal{O}_1]_{n,l}(x_2) \right\rangle$$

[8] Using [F. A. Dolan and H. Osborn, [hep-th/0011040]]

Holography in dS

The second order (one-loop) anomalous dimensions for the double trace operators : $\mathcal{O}_1\square^n\partial^l\mathcal{O}_1$:, : $\mathcal{O}_2\square^n\partial^l\mathcal{O}_2$: and : $\mathcal{O}_1\square^n\partial^l\mathcal{O}_2$: highlight an interesting symmetry between the anomalous dimensions at different spins

$$\gamma^{(2)S}_{n>0,l>0} = -\tfrac{\gamma^2}{l(l+1)}; \qquad \gamma^{(2)A}_{n>0,l>0} = -\frac{\gamma^2}{(2n+l)(2n+l+1)}$$

$$\gamma^{(2)}_{n,2l>0} = \gamma^{(2)S}_{n,2l>0}; \qquad \gamma^{(2)}_{n,2l+1>0} = \gamma^{(2)A}_{n,2l+2}$$

$$\Delta^{S/A} = 2+2n+l+\textstyle\sum_{i=0}^{\infty}\gamma^{(i)S/A}_{n,l}; \qquad \gamma = \frac{\lambda}{16\pi^2}$$

The equations show a degeneracy for the conformal dimensions of these operators for all twists $\Delta_{n,l}-l$, which seems quite remarkable

From the bulk perspective, this is could be a consequence of the symmetry in the EAdS action enforced by the Schwinger-Keldysh formalism and the fact that we take a conformally coupled scalar field. We do not expect this symmetry to hold for general masses

Outlook

We find that the CFT is given by a deformation of a direct product of generalized free fields. However, in contrast to the CFT corresponding to the expansion of the wave function, the cosmological CFT contains three different trajectories of double trace operators due to the mixing of fields with different boundary conditions

$$
\begin{tikzcd}
\boxed{\text{Cosmological correlator CFT}} & & \mathrm{CFT}(\mathcal{O}^S_{n,l}, \mathcal{O}^A_{n,l}, [\mathcal{O}_1\mathcal{O}_2]_{n,l}) & \\
& & \overbrace{\mathrm{GFF}(\mathcal{O}_1)\times \mathrm{GFF}(\mathcal{O}_2)} \arrow[u] \arrow[dl] \arrow[dr] & \\
\boxed{\text{Wave function CFT}} & \mathrm{CFT}_{\Psi[\pi]} & \times & \mathrm{CFT}_{\Psi[\phi]}
\end{tikzcd}
$$

The same analysis can be carried for all conformally coupled scalars although odd and even boundary dimension have different potential

$$V_{\mathrm{odd}} = \phi^{+4} - 6\phi^{+2}\phi^{-2} + \phi^{-4}; \qquad V_{\mathrm{even}} = \phi^{+}\phi^{-}\left(\phi^{+2} - \phi^{-2}\right)$$

Outlook

We have explained that conformally coupled scalar in (rigid) de Sitter and (E)AdS can be evaluated using the techniques from flat space Feynman diagrams.

We have presented a consistent renormalisation scheme and (new) physical quantities can be extracted exactly and compared to the conformal bootstrap results in the Regge limit

Our analytical control of the higher loops Witten diagrams opens to some insight in the by uplift the analysis of unitarity and analyticity for Feynman integrals in flat space

Our formalism allows for consider

- analysis of the unitarity and analyticity in de Sitter by applying techniques from Feynman integrals
- formulate some dS/CFT conjectures
- Consider large-N limit of rigid dS background