Stationary solutions for the equations of 2D fluids

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"Mathematics of geophysical fluid dynamic models of intermediate complexity: qualitative and statistical behaviour"
Reading

Outline

1. The stochastic Navier-Stokes equations
   the stochastic Navier-Stokes eqs with fractional dissipation

2. The damped Euler eqs with stochastic forcing term: invariant measures

3. Eulerian limit à la Kuksin (vanishing viscosity limit): stationary sols for the deterministic unforced Euler eqs

For a homogeneous viscous incompressible fluid

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f + n \\ \nabla \cdot u = 0 \end{cases}$$

$u = u(t, y)$ vector velocity
$p = p(t, y)$ (scalar) pressure
defined for $t \geq 0$, $y \in D \subset \mathbb{R}^d$

$\nu > 0$ kinematic viscosity
f deterministic forcing term
n noise term

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The problem is well posed for $d = 2$

- spatial domain:
  - D bounded and smooth with Dirichlet b.c.
  - $D = [0, L]^2$ with periodic b.c.
  - $D = \mathbb{R}^2$
  - $\vdots$
- noise: smooth enough in space, white in time
  $n = \partial_t W$

Fix D smooth and bounded, assume Dirichlet b.c.
$\mathcal{C}^\infty = \{u \in [C_0^\infty(D)]^2 : \text{ div } u = 0\}$
$H = \overline{\mathcal{C}^\infty}^{L^2}$
$V = H \cap [H_0^1(D)]^2$
V' dual of V

Q covariance of the Wiener process

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Theorem (solutions)

Let $u_0 \in H$, $f \in L^2(0,T;V')$, Q trace class operator in H. Then there exists a unique solution u to the Navier-Stokes equations. The paths are a.s. in $C([0,T];H) \cap L^2(0,T;V)$.

As soon as a unique solution exists for any $u_0 \equiv x \in H$, one define the Markov semigroup $P_t : B_b(H) \to B_b(H)$

$$P_t\phi(x) = \mathbb{E}[\phi(u(t;x))]$$

Its dual acts on measures μ defined on Borel subsets of H:

$$\int \phi dP_t^* \mu = \int P_t \phi d\mu$$

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A probability measure μ is said to be an invariant measure if

$$P_t^*\mu = \mu \qquad \forall t \geq 0$$

i.e.

$$\int P_t\phi d\mu = \int \phi d\mu \qquad \forall t \geq 0, \phi \in B_b(H).$$

Ergodicity

Let $\{e_j\}$ be the complete orthonormal system of the eigenfunctions of Δ: $-\Delta e_j = \lambda_j e_j$. Set

$$W(t) = \sum_{j=1}^{\infty} \sigma_j w_j(t) e_j$$

for a sequence of i.i.d. real Wiener processes $\{w_j\}_j$.

Assume the noise has one of the following properties

- $\sigma_j \neq 0$ for a finite number of j, suitably chosen
- $\sigma_j \neq 0$ for all j and $\sigma_j \sim \lambda_j^{-a}$ for some $a > \frac{3}{8}$

Theorem (invariant measures)

On the torus...
There exists a unique invariant measure for the Navier-Stokes equations.

f can be added....

The unique invariant measure μ is ergodic: for μ-a.e. x

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \phi(u(t;x))\ dt = \int \phi\ d\mu \qquad a.s.$$

for any $\phi \in L^1(\mu)$

$\vdots$

For $\alpha > 0$:

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\alpha} u + (u \cdot \nabla)u + \nabla p = n \\ \nabla \cdot u = 0 \end{cases}$$

$0 < \alpha < 1$: hypoviscous Navier-Stokes eqs

$\alpha > 1$: hyperviscous Navier-Stokes eqs

Hypoviscous is more difficult but solutions smooth to an arbitrary degree of regularity after an arbitrarily short time.

Similar results (on the torus): for solutions and for invariant measures

Theorem (Constantin, Glatt-Holtz, Vicol. 2014)

There exists an $N = N(\alpha, Q)$ such that if $\sigma_j \neq 0$ for all $j = 1, \ldots, N$, there exists a unique and ergodic invariant measure.

The viscosity is fixed.

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are a perturbation of the unforced Euler equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases}$$

for which invariants are known (the energy , the enstrophy, ...). Adding a dissipative term and a stochastic forcing term: a balance is reached and therefore there exist invariant measures.

$$\begin{cases} \partial_t u + \nu(-\Delta)^\alpha u + (u \cdot \nabla) u + \nabla p = f + n \\ \nabla \cdot u = 0 \end{cases}$$

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Dissipation of order 0 $\rightsquigarrow$ damping

$$\begin{cases} \partial_t u + \gamma u + (u \cdot \nabla) u + \nabla p = n \\ \nabla \cdot u = 0 \end{cases}$$

damped Euler equations with stochastic force

In vorticity form:
define the vorticity $\xi = \nabla^{\perp} \cdot u$

$$d\xi + [\gamma\xi + u \cdot \nabla\xi]\; dt = dW$$

Existence of stationary solutions has been proved when $\gamma > 0$ (see Bessaih 2008).

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Existence of invariant measures for the stochastic damped Euler equation is obtained by means of the Bogoliubov-Krylov's technique, suitably modified so to work in space L^∞ (not a Polish space).

Bogoliubov-Krylov's technique

The "classical" version (see Da Prato-Zabczyk) is the following.

Define the Markov semigroup $P_t : B_b(X) \to B_b(X)$ as

$$P_t\phi(\chi) = \mathbb{E}[\phi(\xi^\chi(t))]$$

Proposition

Let X be a separable Banach space. If

- *(Feller property)* $\quad P_t : C_b(X) \to C_b(X)$
- *the sequence of measures* $\mu_n = \frac{1}{n}\int_0^n P_s^*\delta_0 ds$ *is tight in* X

then there exists a measure μ on the Borelian subsets of X which is invariant, that is

$$\int P_t\phi \, d\mu = \int \phi \, d\mu \qquad \forall t \geq 0, \phi \in C_b(X).$$

Work in L^∞ equipped with the bounded weak$\star$ topology or the weak$\star$ topology.

[the weak$\star$ topology is the weakest topology for which the mappings $L^\infty \ni \xi \mapsto \langle \xi, g \rangle \in \mathbb{R}$ are continuous for any $g \in L^1$

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Let us look at the "ingredients" of Bogoliubov-Krylov's techinque:

Feller+tightness

Tightness

A nice issue with the weak topology has been introduced by Maslowski and Seidler (1999): **use weak topologies** to get tightness. It is easier than with strong topology. But they worked in a separable Hilbert space!
For the equation

$$d\xi + (\gamma\xi + u \cdot \nabla\xi)\ dt = dW$$

(similar structure as a transport eq: easy to get estimates in L^p and L^∞)

we can prove uniform L^∞-bounds in probability (for any $\gamma > 0$).

Proposition (w$\star$ tightness)

Let $\gamma > 0$. Then, for any $\epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$\inf_{t\geq 0} \mathbb{P}\{\|\xi^0(t)\|_{L^\infty} \leq R_\epsilon\} \geq 1-\epsilon.$$

Since the balls in L^∞ are compact for the weak$\star$ topology (and for the bounded weak$\star$ topology), we get **tightness** of the sequence of measures

$$\mu_n = \frac{1}{n}\int_0^n \mathcal{L}(\xi^0(s))ds$$

with respect to the weak$\star$ topology.

Looking for the transition semigroup

We can prove a weak form of continuous dependence on the initial data: $\chi \mapsto \xi^\chi(t)$ is sequentially continuous.

Proposition

Let $\gamma \geq 0$.
Given a sequence $\{\chi^n\}_n \subset L^\infty$ which converges weak$\star$ in L^∞ to $\chi \in L^\infty$, we have that, $\mathbb{P}$-a.s., for every $t > 0$ the sequence $\left\{\xi^{\chi^n}(t)\right\}_n$ converges weak$\star$ in L^∞ to $\xi^\chi(t)$.

Therefore we have a "weak Feller" property for the operator P_t defined as

$$P_t\phi(\chi) = \mathbb{E}[\phi(\xi^\chi(t))].$$

$$\chi \mapsto \xi^\chi(t) \mapsto \phi(\xi^\chi(t)) \mapsto \mathbb{E}[\phi(\xi^\chi(t))] = P_t\phi(\chi)$$

Proposition

The operator P_t is sequentially weak$\star$ Feller in L^∞, that is

$$P_t : SC_b(L^\infty, \mathcal{T}_{w\star}) \to SC_b(L^\infty, \mathcal{T}_{w\star}) \tag{1}$$

for any $t \geq 0$.

Since $C(L^\infty, \mathcal{T}_{bw\star}) = SC(L^\infty, \mathcal{T}_{w\star})$, this is equivalent to be Feller with respect to the bounded weak$\star$ topology

$$P_t : C_b(L^\infty, \mathcal{T}_{bw\star}) \to C_b(L^\infty, \mathcal{T}_{bw\star}) \tag{2}$$

Markov property

For every $\phi \in SC_b(L^\infty, \mathcal{T}_{w\star})$, $\chi \in L^\infty$ and $t, s > 0$

$$\mathbb{E}\left[\phi\left(\xi^\chi(t+s)\right)|\mathcal{F}_t\right] = \left(P_s\phi\right)\left(\xi^\chi(t)\right) \qquad \mathbb{P}-a.s. \tag{3}$$

obtained by an approximation (first, working in $W^{1,4}$, which is separable; then we use that $W^{1,4}$ is dense in L^∞ with respect to the weak$\star$ topology $\mathcal{T}_{w\star}$).

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obtained by an approximation (first, working in $W^{1,4}$, which is separable; then we use that $W^{1,4}$ is dense in L^∞ with respect to the weak$\star$ topology $\mathcal{T}_{w\star}$).
Taking the expectation, we get

$$\mathbb{E}\left[\phi\left(\xi^\chi(t+s)\right)\right] = \mathbb{E}\left[(P_s\phi)\left(\xi^\chi(t)\right)\right]$$

which can be rewritten as (semigroup property)

$$(P_{t+s}\phi)(\chi) = \left(P_t(P_s\phi)\right)(\chi).$$

Hence we have $P_{t+s} = P_t P_s$ on $SC_b(L^\infty, \mathcal{T}_{w\star})$.

Summing up: we have

- a Markov semigroup $\{P_t\}_t$ acting on $C_b(L^\infty, \mathcal{T}_{bw\star})$
- a tight sequence of measures μ_n with respect to the bounded weak$\star$ topology $\mathcal{T}_{bw\star}$

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- a Markov semigroup $\{P_t\}_t$ acting on $C_b(L^\infty, \mathcal{T}_{bw\star})$
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We are ready to use the Bogoliubov-Krylov's technique to get existence of invariant measures.
We apply Prokhorov's theorem in the version given by Jakubowski (1997) so to work in nonmetric spaces.

Existence of invariant measures

Theorem (Bessaih, F)

Let $\gamma > 0$.
Then there exists at least one invariant measure μ for the stochastic damped Euler equation.

This is a measure on the Borel subsets $\mathcal{B}(\mathcal{T}_{bw\star}) = \mathcal{B}(\mathcal{T}_{w\star})$ such that

$$\int P_t\phi \, d\mu = \int \phi \, d\mu$$

for all $t \geq 0$ and $\phi \in C_b(L^\infty, \mathcal{T}_{bw\star})$.

Kuksin in a series of papers considers the vanishing viscosity limit of

$$\begin{cases} \partial_t u + [-\nu\Delta u + (u\cdot\nabla)u + \nabla p]\ dt = \sqrt{\nu}dW \\ \nabla\cdot u = 0 \end{cases}$$

In *J. Stat. Phys.* (2004) he proves that given any family of stationary solutions $\{u^\nu\}_{\nu>0}$ there exists a subsequence (with $\nu_n \to 0$) such that u^{ν_n} converges in distribution to a non-trivial stationary process U, solving the Euler eq. (on the torus)

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Crucial estimates for the viscous eqs are

$$\frac{d}{dt}\mathbb{E}\|u^\nu(t)\|^2_{L^2} + 2\nu \mathbb{E}\|\nabla u^\nu(t)\|^2_{L^2} = \nu q$$

$$\frac{d}{dt}\mathbb{E}\|\xi^\nu(t)\|^2_{L^2} + 2\nu \mathbb{E}\|\nabla \xi^\nu(t)\|^2_{L^2} = \nu q^\perp$$

Therefore from

$$\frac{d}{dt}\mathbb{E}\|u^\nu(t)\|^2_{L^2} + 2\nu\mathbb{E}\|\nabla u^\nu(t)\|^2_{L^2} = \nu q$$

$$\frac{d}{dt}\mathbb{E}\|\xi^\nu(t)\|^2_{L^2} + 2\nu\mathbb{E}\|\nabla \xi^\nu(t)\|^2_{L^2} = \nu q^\perp$$

(here $\xi = \nabla^\perp \cdot u$ is the vorticity) we get for a stationary solution

$$2\mathbb{E}\|\nabla u^\nu(t)\|^2_{L^2} = q$$

$$2\mathbb{E}\|\nabla \xi^\nu(t)\|^2_{L^2} = q^\perp$$

$\rightsquigarrow$ tightness (uniform estimates for $\nu > 0$)

The limit U has energy and enstrophy which are time-independent random constants, and U depends on the covariance of the noise, i.e. q and $q^{\perp}$:

$$\frac{q^2}{2q^{\perp}} \leq \mathbb{E}\|U(t)\|^2_{H^0} \leq \frac{1}{2}q$$

$$\mathbb{E}\|U(t)\|^2_{H^1} = \frac{1}{2}q$$

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Does the limit depend on the Laplace operator (i.e. on the approximating eq)? What happens for

$$\partial_t u + [\nu(-\Delta)^{\alpha} u + (u \cdot \nabla)u + \nabla p]\ dt = \sqrt{\nu} dW$$

when $\alpha \neq 1$?

Following the idea of Kuksin we get that the hypo- or hyper-viscous Navier-Stokes eqs has an Eulerian limit

Theorem (hypoviscous eq)

For any $0 < \alpha < 1$ there exists a stationary process $U^{(p)} \in L^2_{loc}(\mathbb{R}_+; H^{1+\alpha})$, $\frac{dU^{(\alpha)}}{dt} \in L^1_{loc}(\mathbb{R}_+; H^{\alpha})$ solving the Euler equation. Moreover

$$\frac{1}{2}q\left(\frac{q}{q^\perp}\right)^\alpha \leq \mathbb{E}\|U^{(\alpha)}(t)\|^2_{H^0} \leq \frac{1}{2}q \tag{4}$$

$$\mathbb{E}\|U^{(\alpha)}(t)\|^2_{H^\alpha} = \frac{1}{2}q \tag{5}$$

$$\mathbb{E}\|U^{(\alpha)}(t)\|^2_{H^{\alpha+1}} \leq \frac{1}{2}q^\perp \tag{6}$$

Similar result (even stronger) for the hyperviscous eqs.

Moreover, if the noise mixes the dynamics (the noise acts on at least four particular directions), then the Eulerian limits obtained for different α and $\tilde{\alpha}$ are different

since if they were equal $U^{(\alpha)} = U^{(\tilde{\alpha})} = U$, then

$$\mathbb{E}\|U(t)\|^2_{H^\alpha} = \frac{1}{2}q = \mathbb{E}\|U(t)\|^2_{H^{\tilde{\alpha}}}$$

but this is true only if $U = \sum_{k\in\mathbb{Z}^2} U_k e_k$ has at most four Fourier modes $\{U_k\}_{|k|=1}$.

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CONCLUSION:

A quite large number of stationary solutions exist for the deterministic unforced 2D Euler eqs (on the torus).

They depend on the intensity of the noise (q and $q^\perp$) and on the power of the Laplacian $\alpha > 0$.

They are not the vanishing solution $U = 0$.

THANK YOU