Motion, Monodromy, and Asymptotics

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@monsoon0

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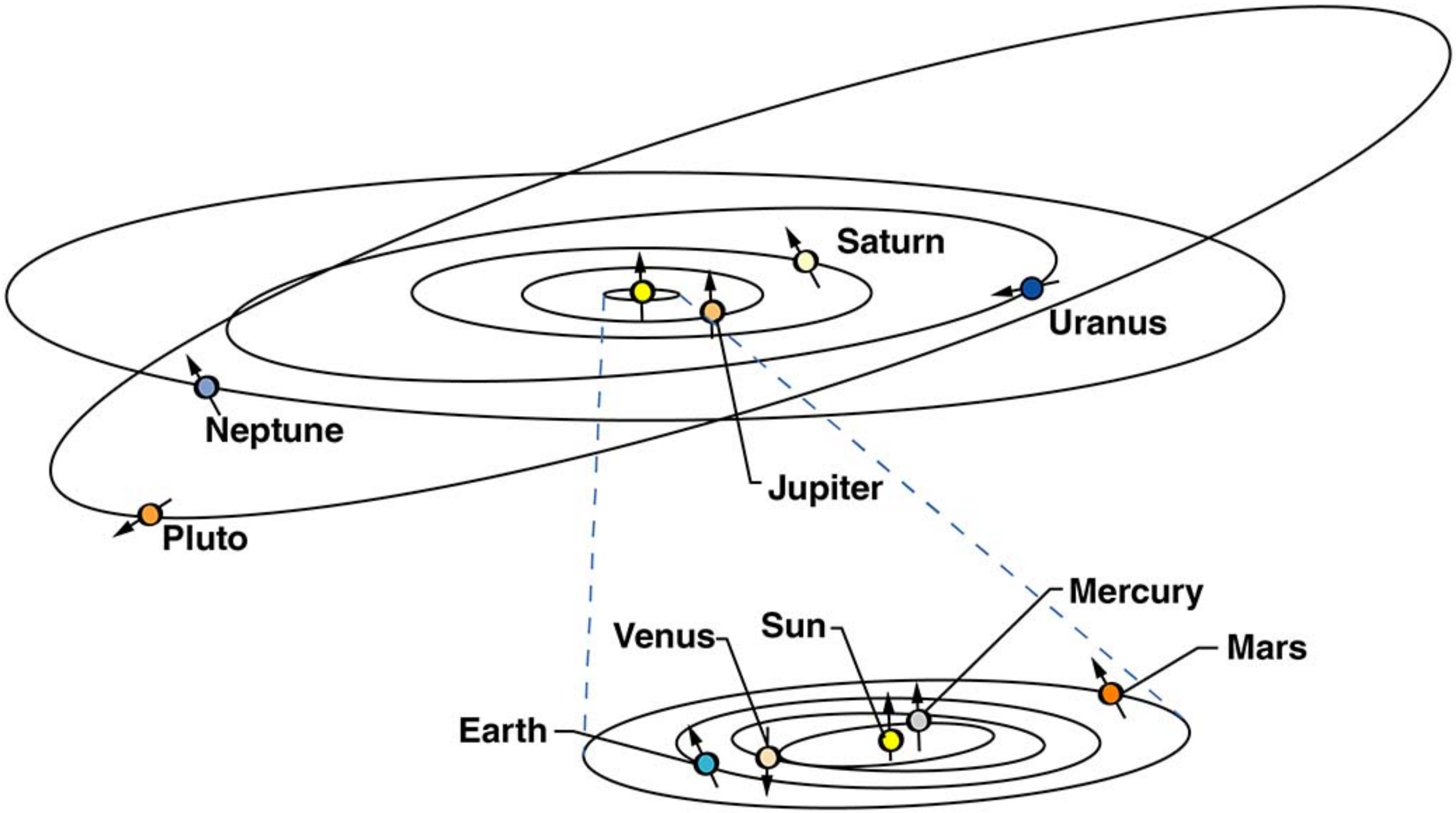
Grace Cossington Smith "*The bridge in-curve*" 1930, National Gallery of Victoria, Australia

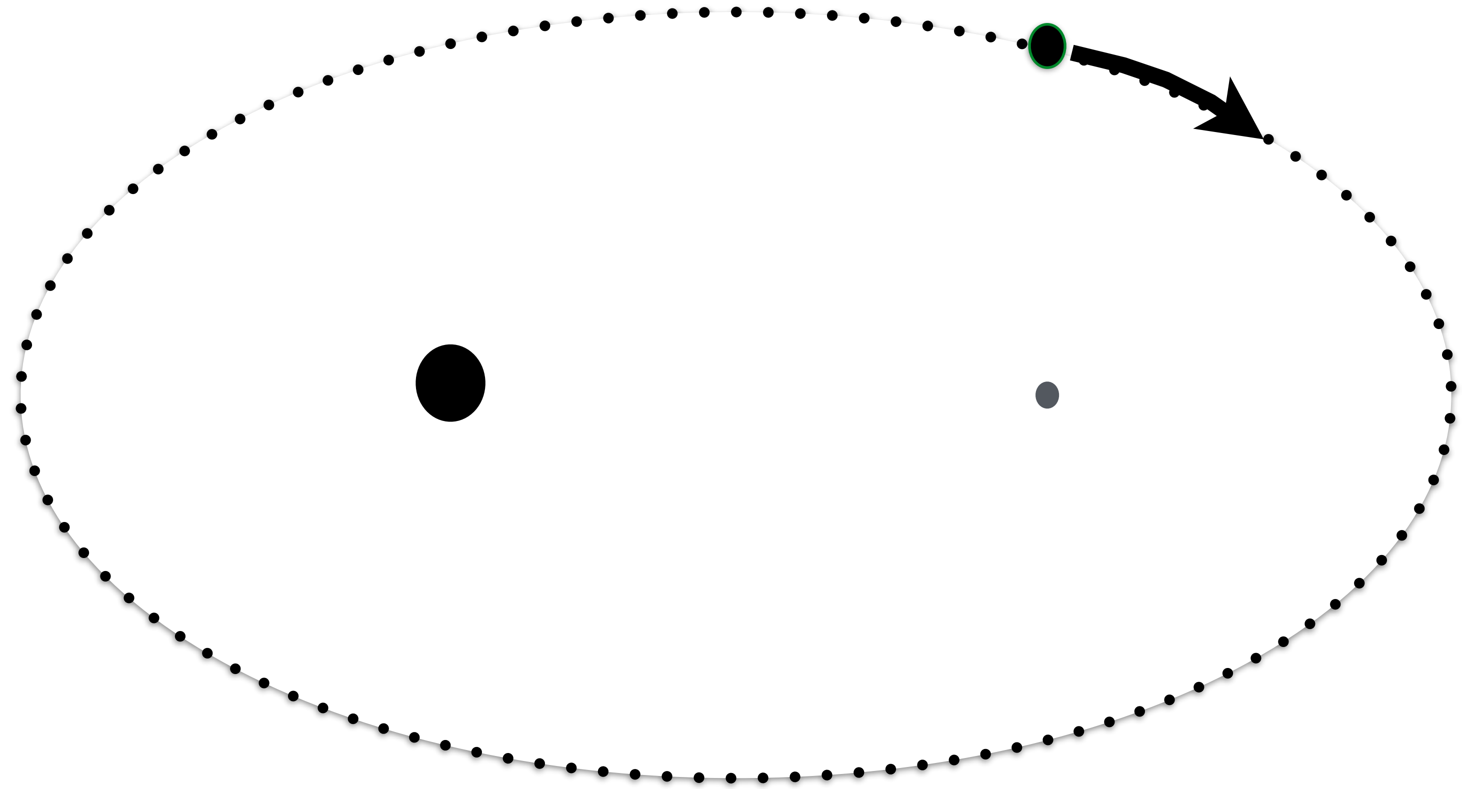
Motion



Wikipedia

Elliptic orbits

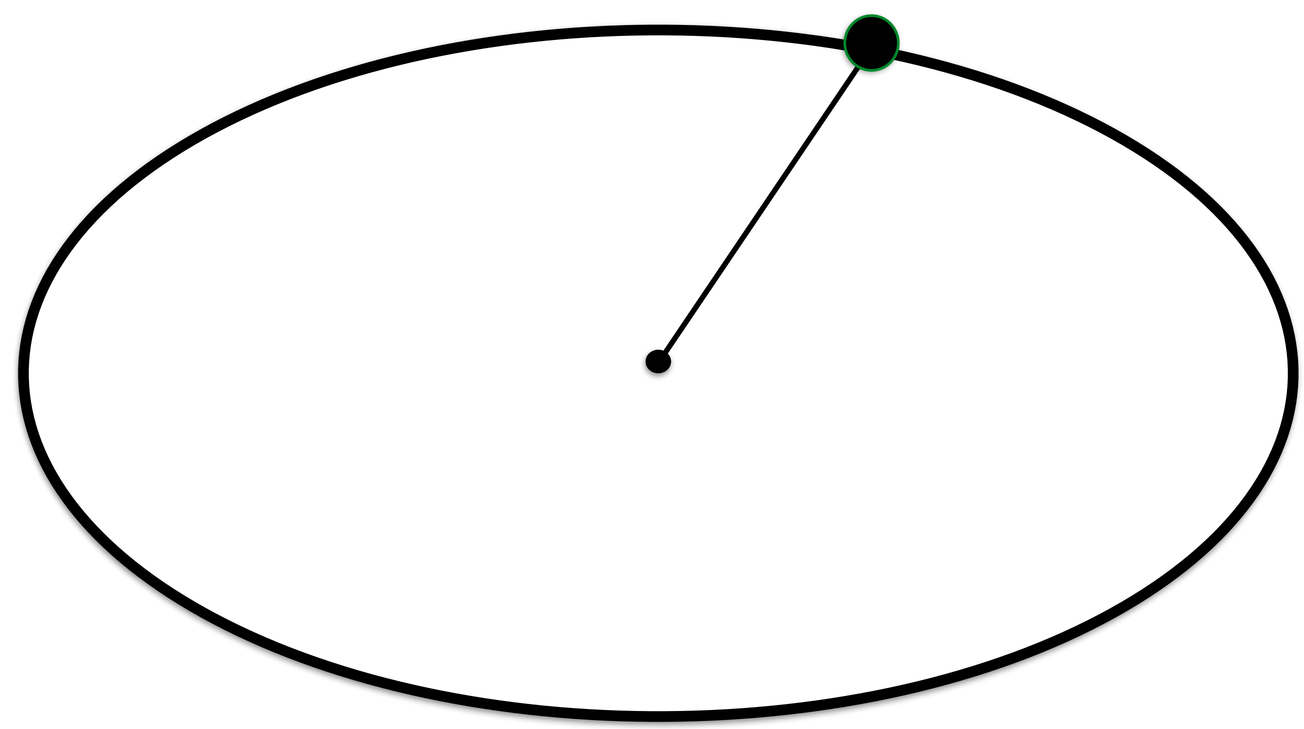
Kepler's first law: the orbit of every planet is an ellipse with the Sun at one of the two foci.



Polynomials

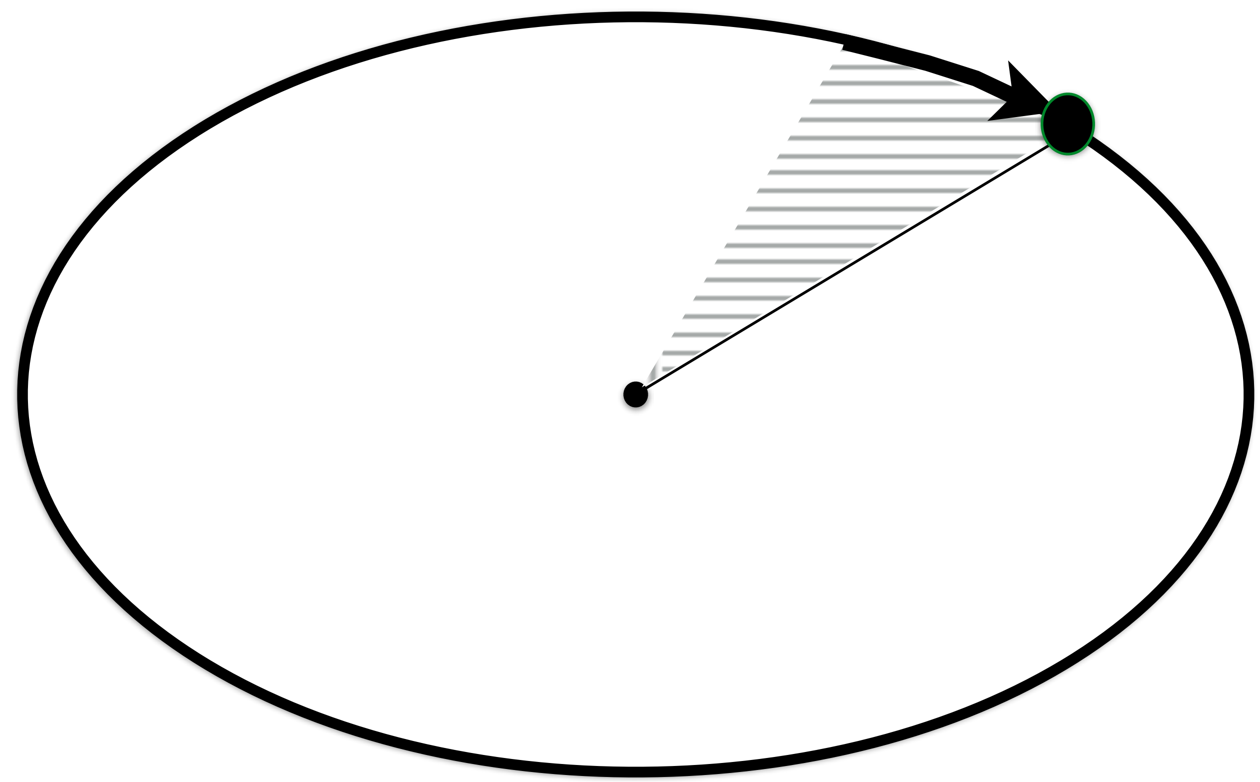
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

The position (x, y) of a planet moving on an elliptical orbit is given by a polynomial of x and y.



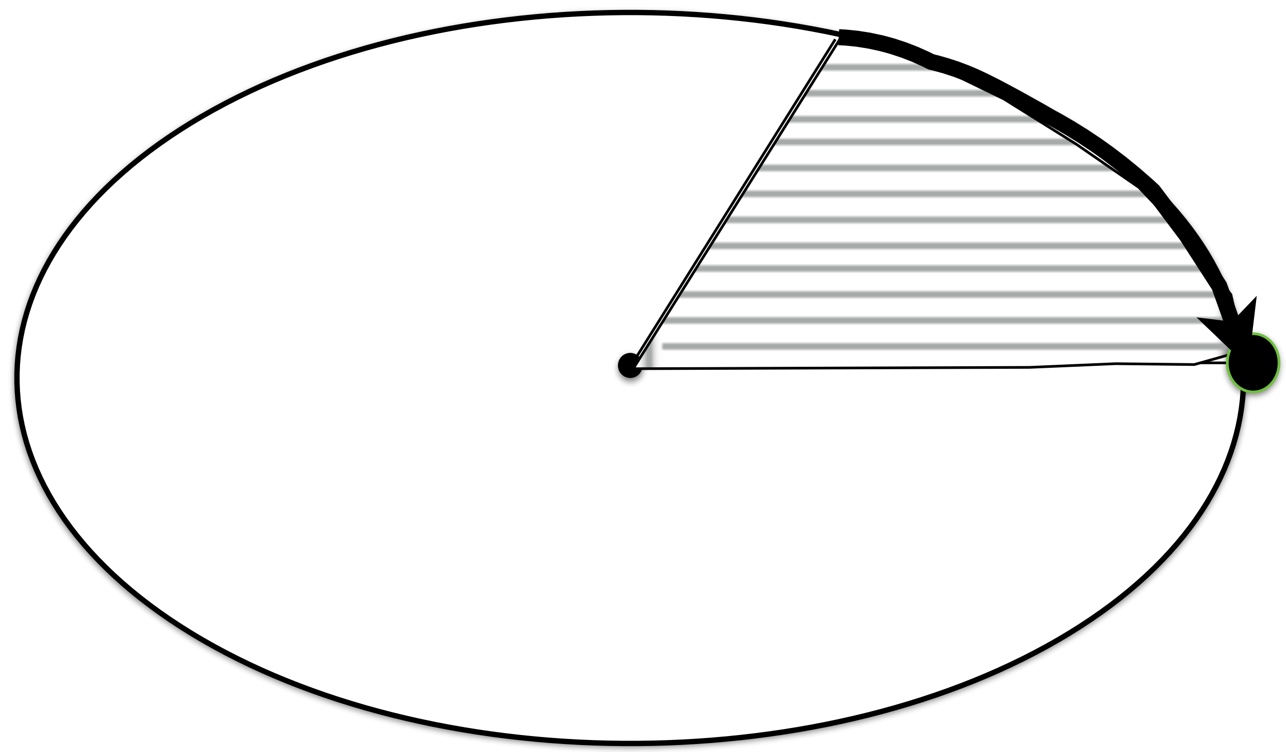
Area and arc length

The area and arc length swept out by the ray to the planet are also functions of (x, y).



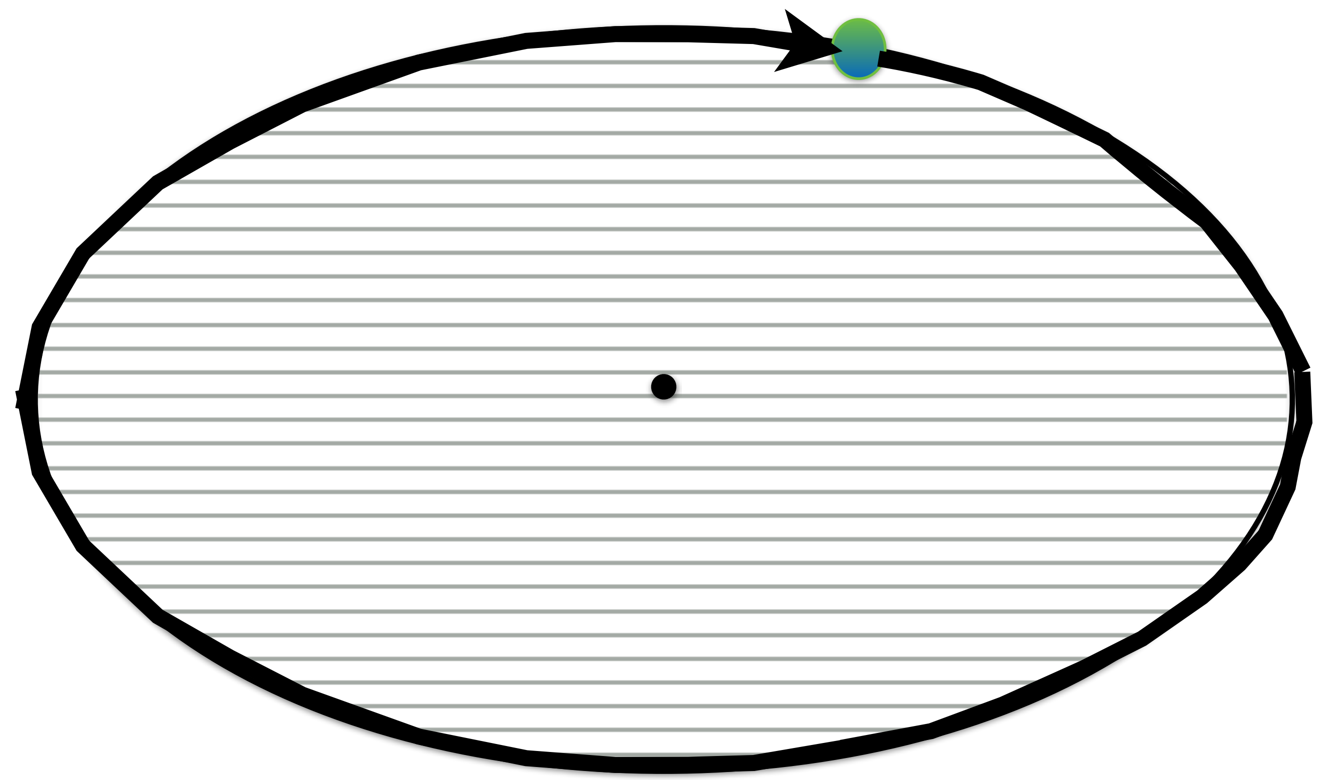
Newton's question

So Newton asked, in 1687, whether the arc length and area swept out by the planet are also solutions of polynomial equations.



Newton's answer I

As (x, y) moves along the orbit with time t and returns to the starting point, the ray to the planet has swept out the whole area bounded by the orbit.



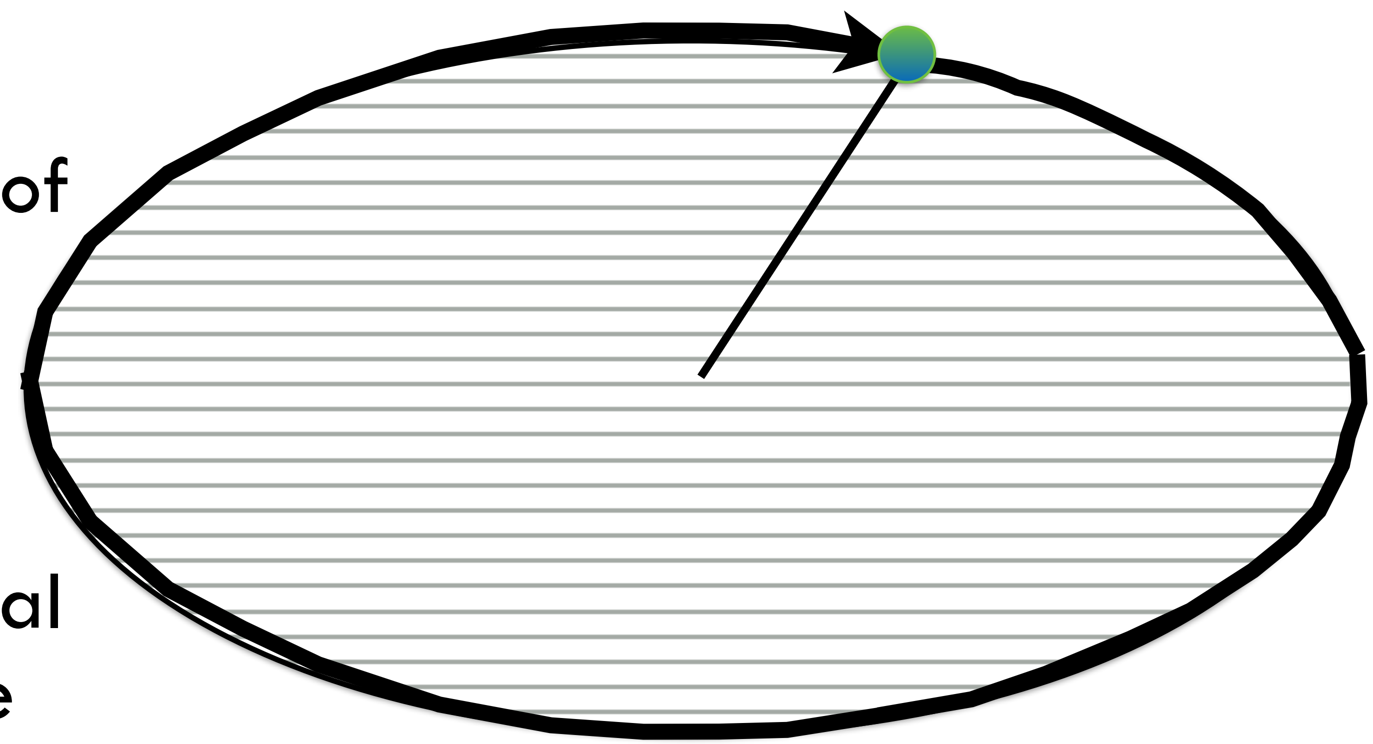
Newton's answer II

The area increases again after another orbit, and again and again…

So, it takes an infinite number of values at each point (x, y)

The same is true of the arc length.

But, the solution of a polynomial equation can only take a finite number of values (determined by its degree).



So the area and arc length cannot be solutions of a polynomial equation $\Rightarrow$ They are transcendental functions.

The arc length

The length along a small arc of the ellipse is

$$ds^2 = dx^2 + dy^2 = \left(a^2 \cos^2(\theta) + b^2 \sin^2(\theta)\right) d\theta^2$$

$$\Rightarrow \quad s(\theta) = a \int_0^\theta \sqrt{1 - k^2 \sin^2(\theta)}\, d\theta$$

Euler, 1738

where $k^2 = 1 - b^2/a^2$

This is an elliptic integral (of the second kind). Legendre classified such integrals into three kinds in 1811. The integral of the first kind is

$$F(\theta, k) = \int_0^\theta \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}$$

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Inversion leads to *elliptic functions:* $\mathrm{sn}(t; k), \ldots$

Abel 1825

Weierstrass form

$$F(w,k) = \int_w^\infty \frac{du}{\sqrt{4u^3 - g_2 u + g_3}}$$

Inverse function: $\Rightarrow \wp(x; g_2, g_3)$

Weierstrass 1840s

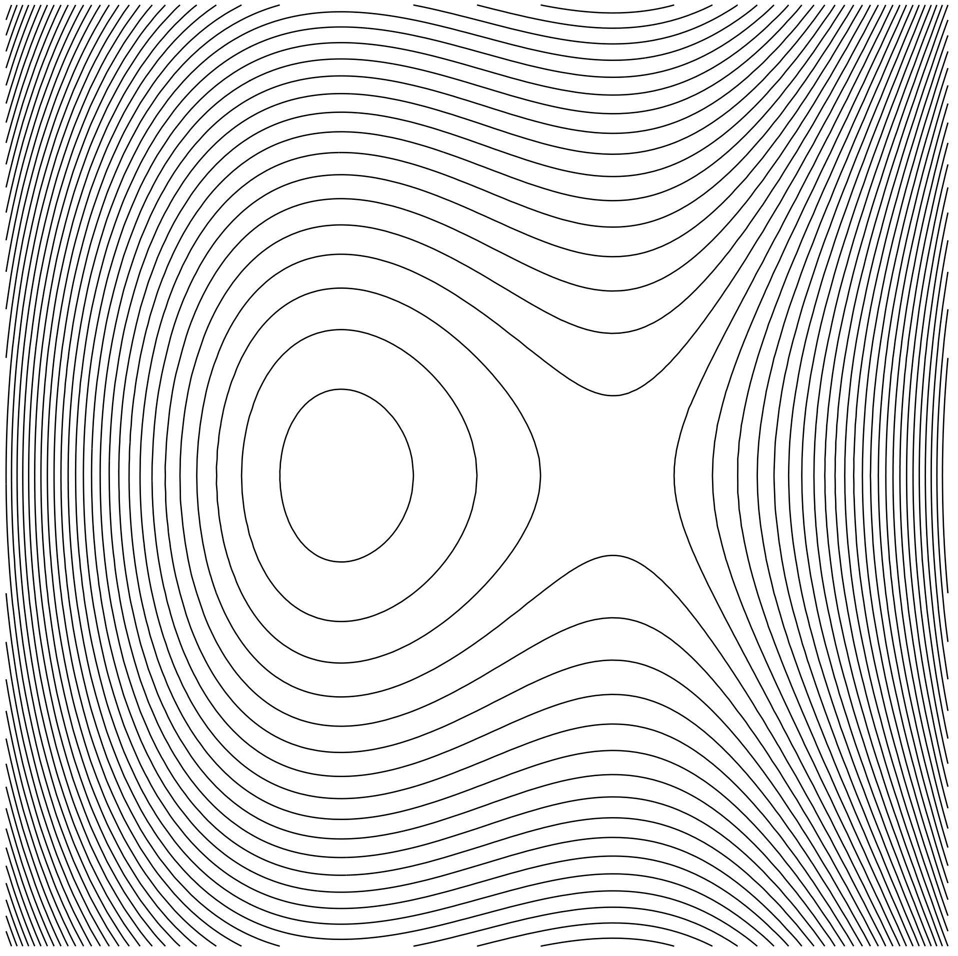
which satisfies differential equations:

$$w_x^2 = 4w^3 - g_2 w + g_3$$

$$w_{xx} = 6w^2 - g_2$$

Weierstrass model

$$y^2 - 4x^3 + g_2 x - g_3 = 0$$



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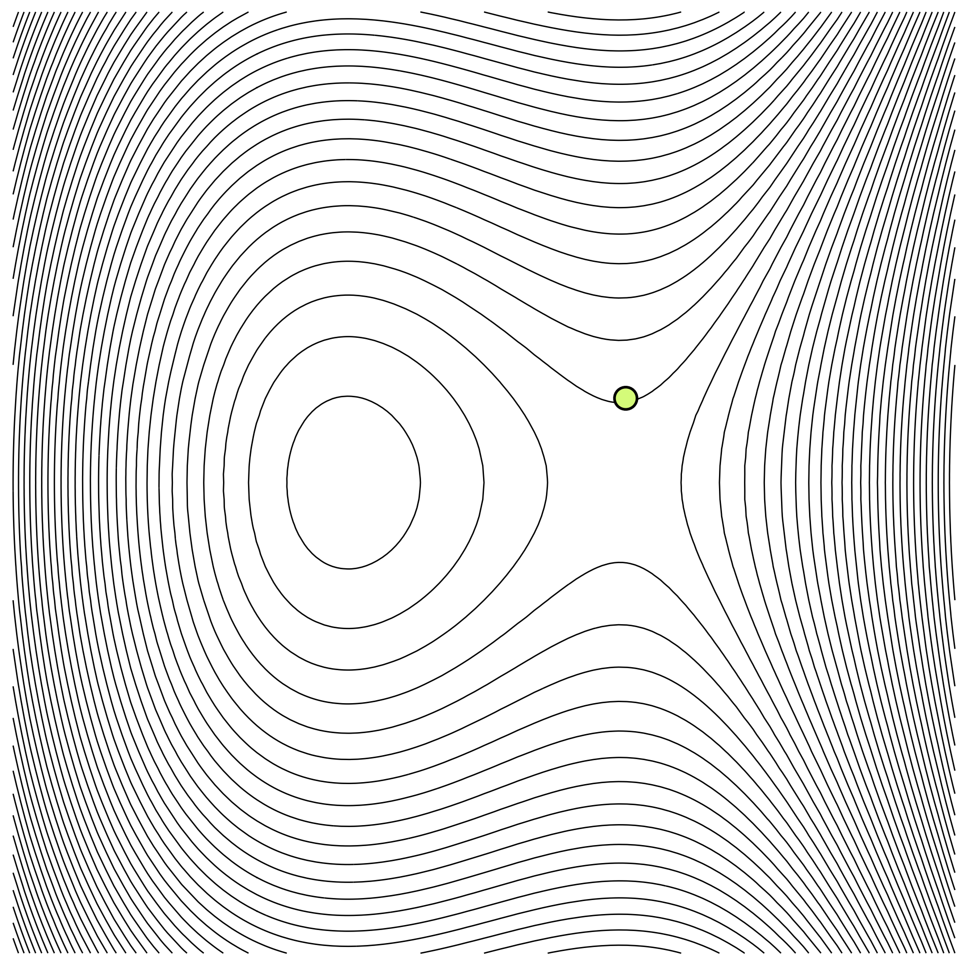

For fixed g_2, each curve is parametrised by g_3, with points given by $\big(\wp(s - s_0), \wp'(s - s_0)\big)$.

Weierstrass model

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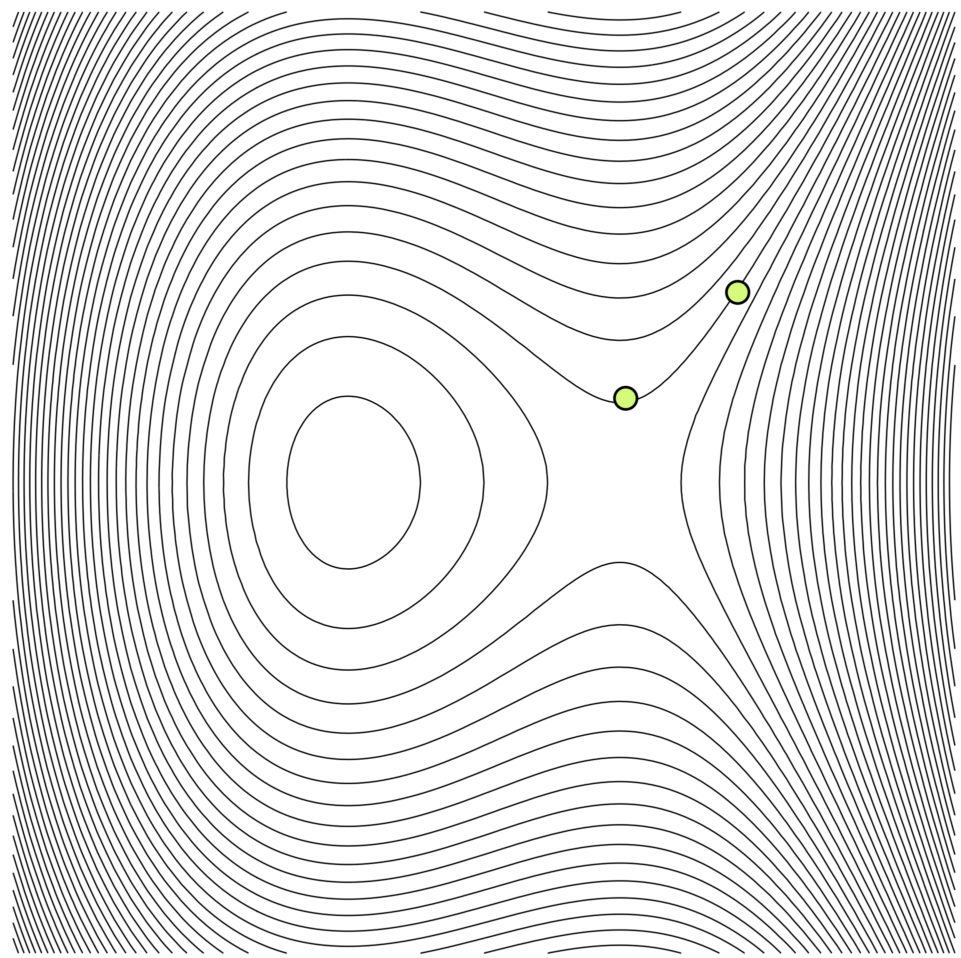

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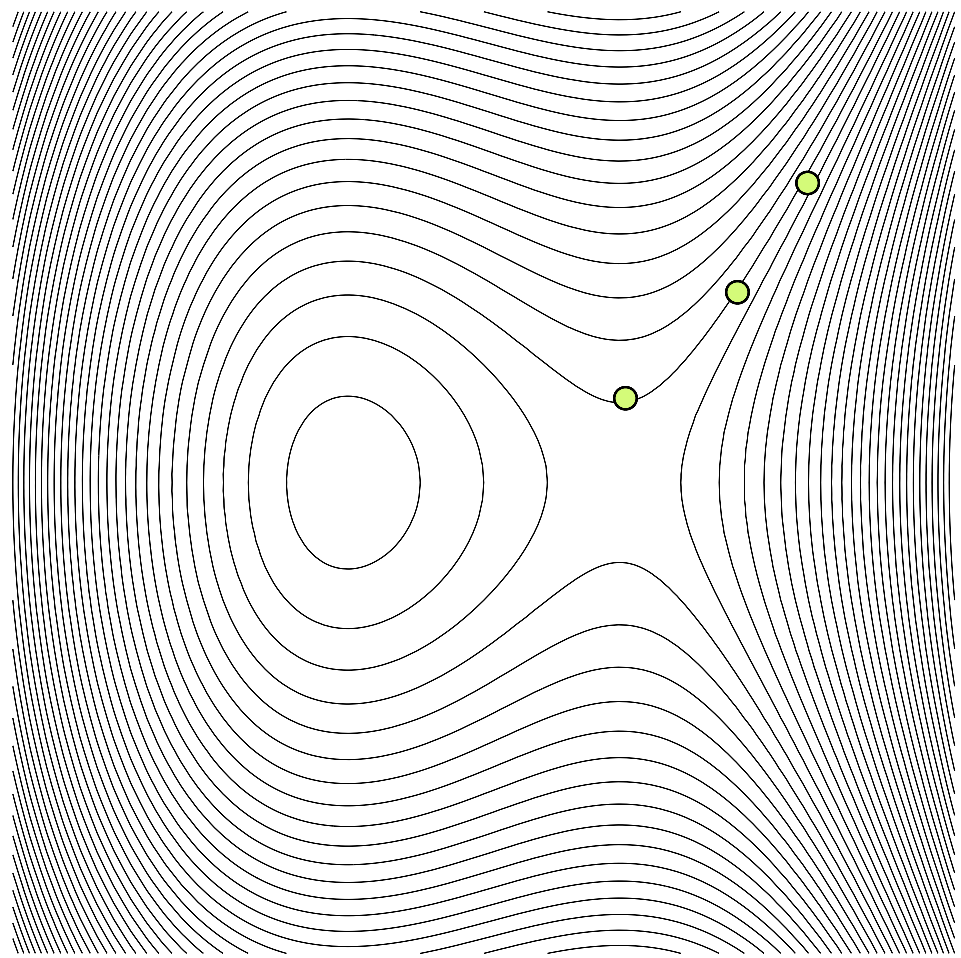

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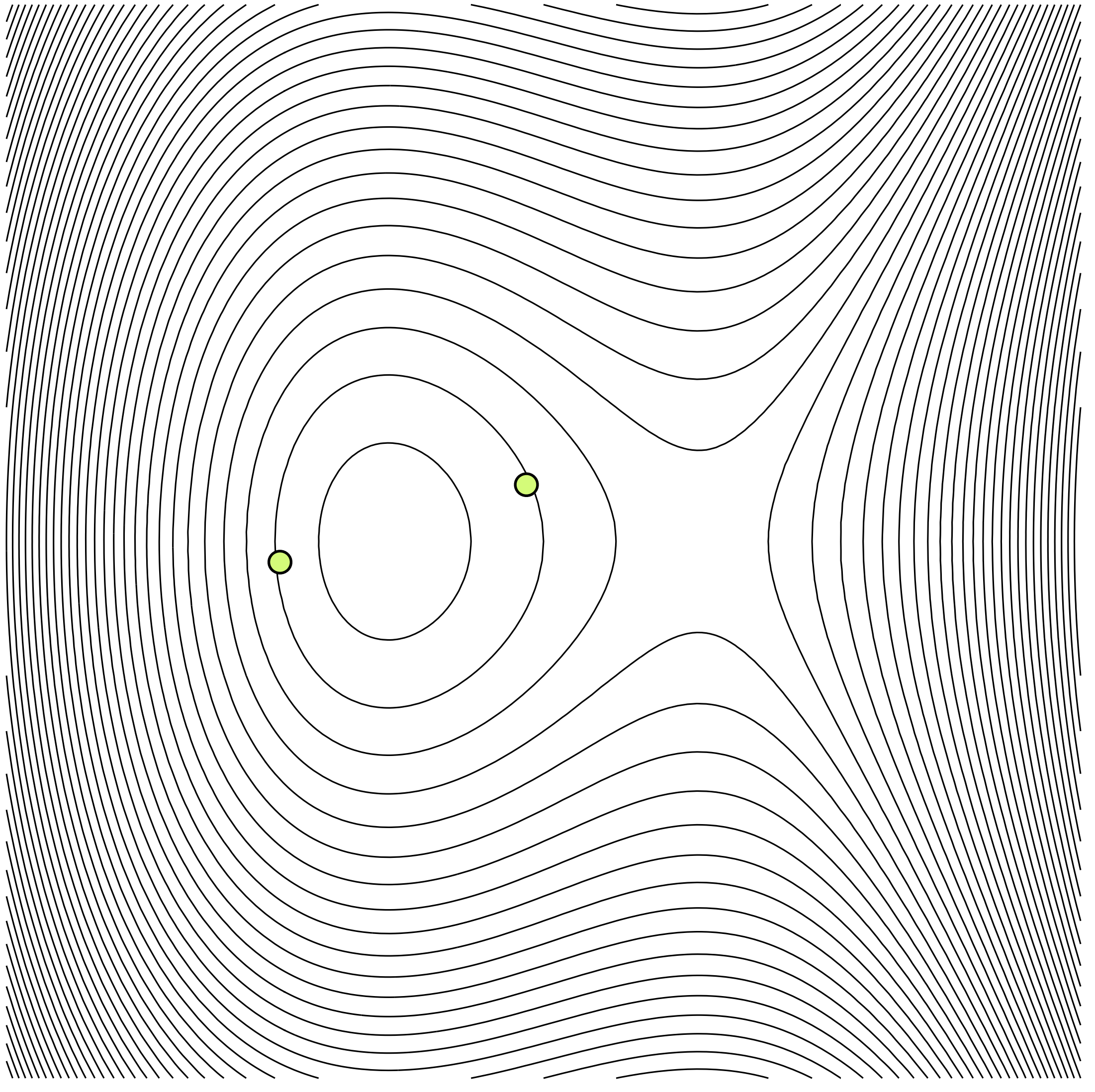
Discrete motion



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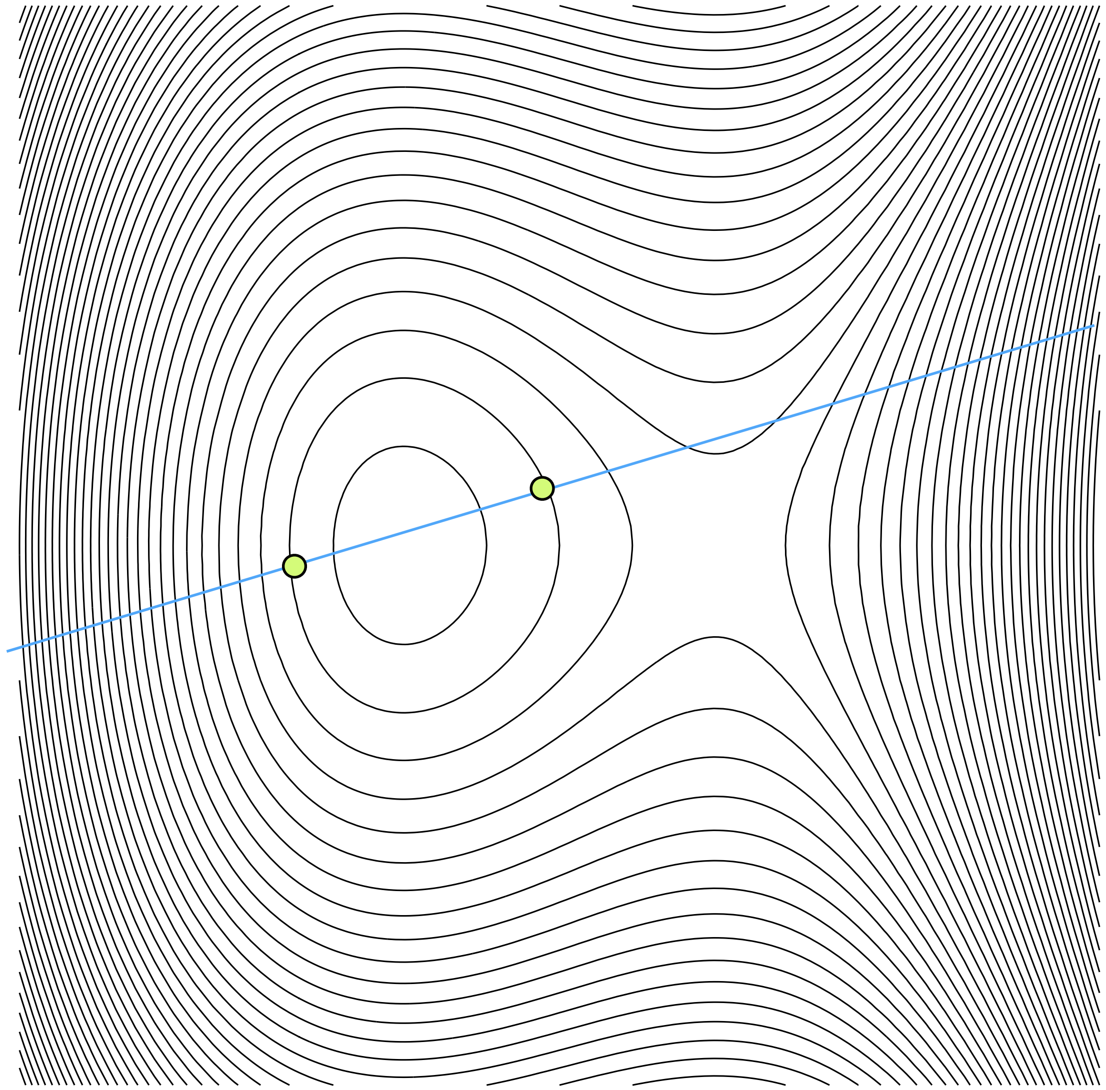

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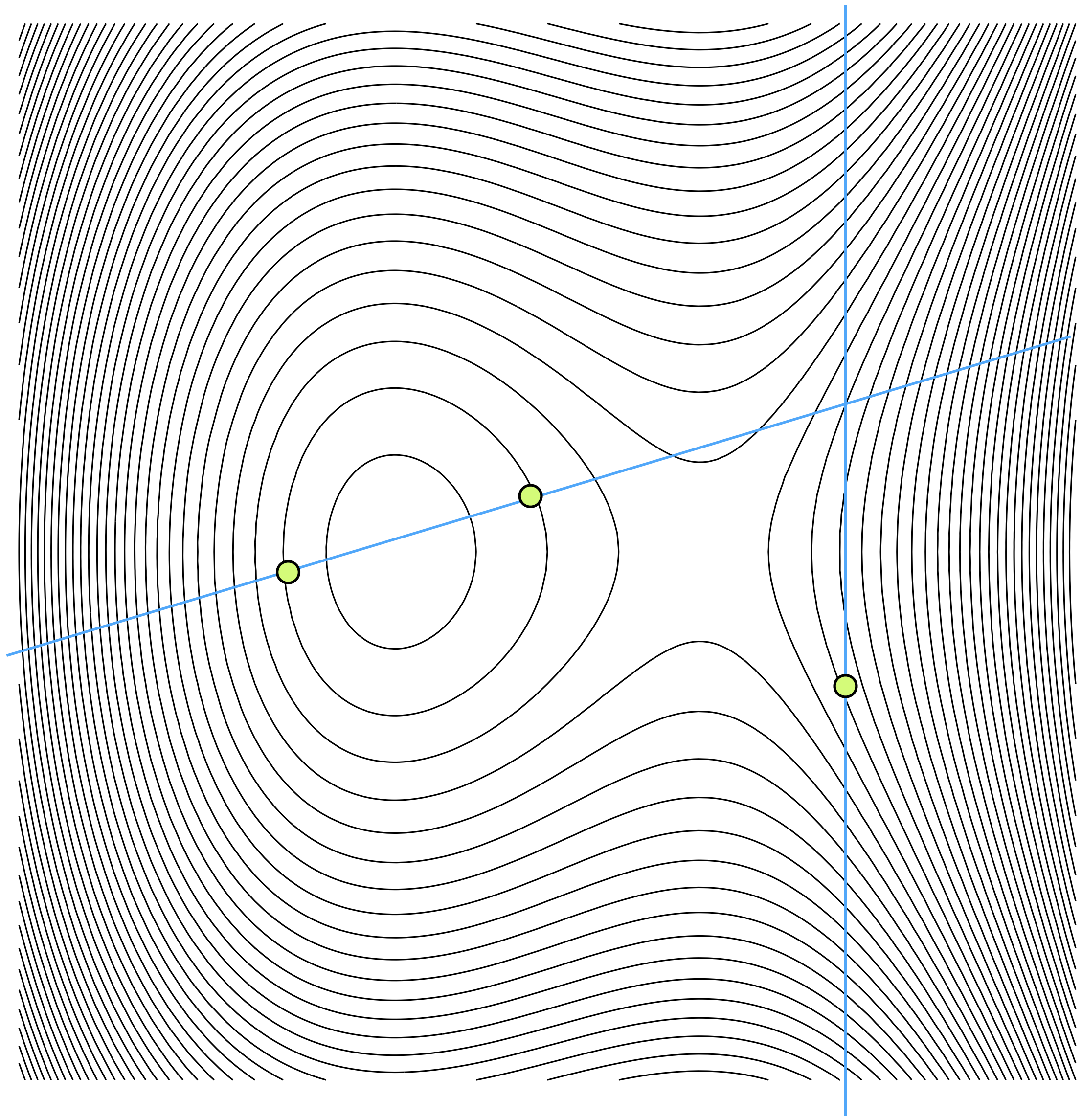

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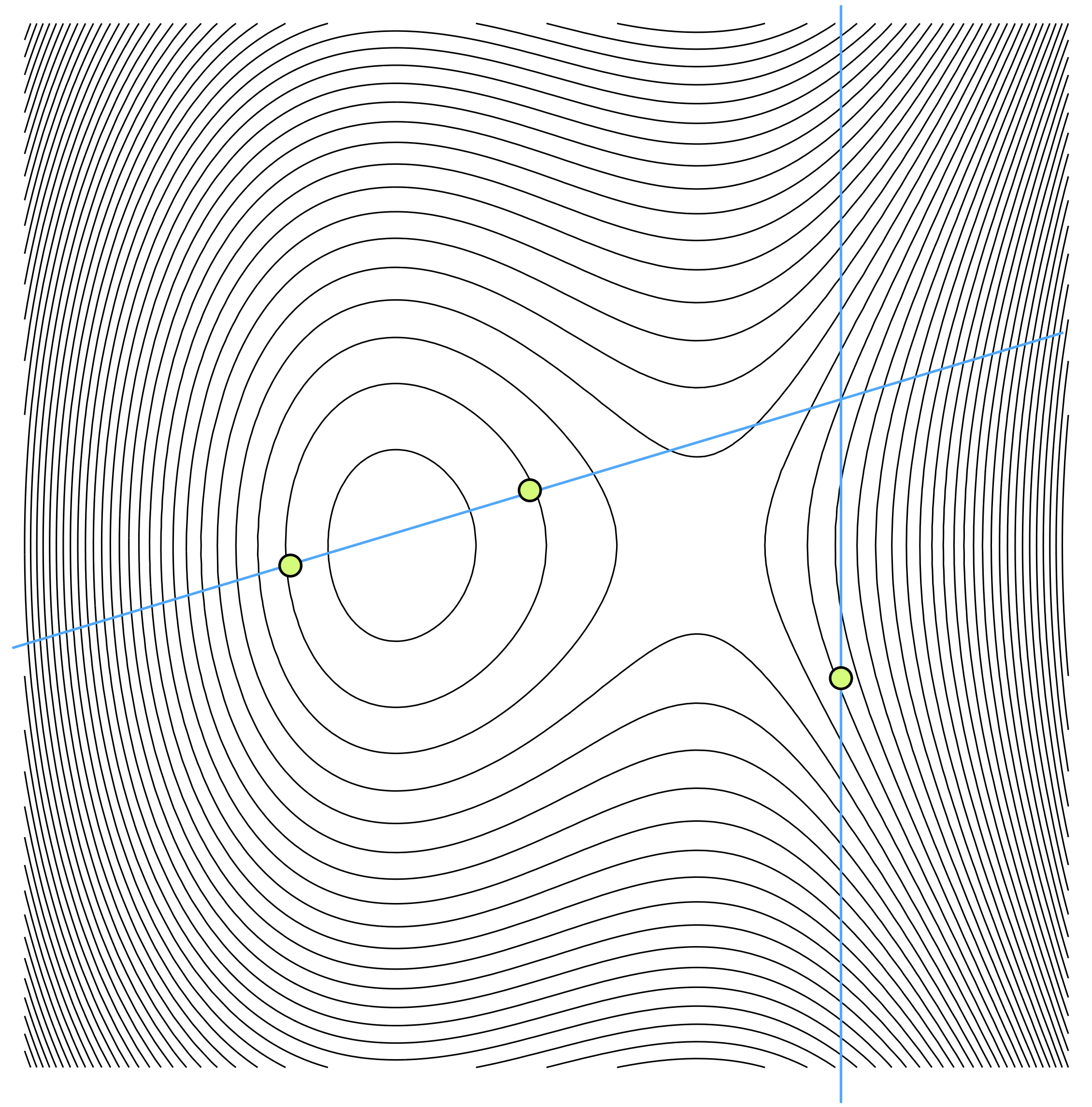


Elliptic functions satisfy an addition theorem.

Cubic curves

$$y^2 = 4x^3 - g_2 x + g_3$$

Newton classified all irreducible cubic curves in an unpublished manuscript, and defined functions through "Puiseux" series, e.g.,

Newton, 1676

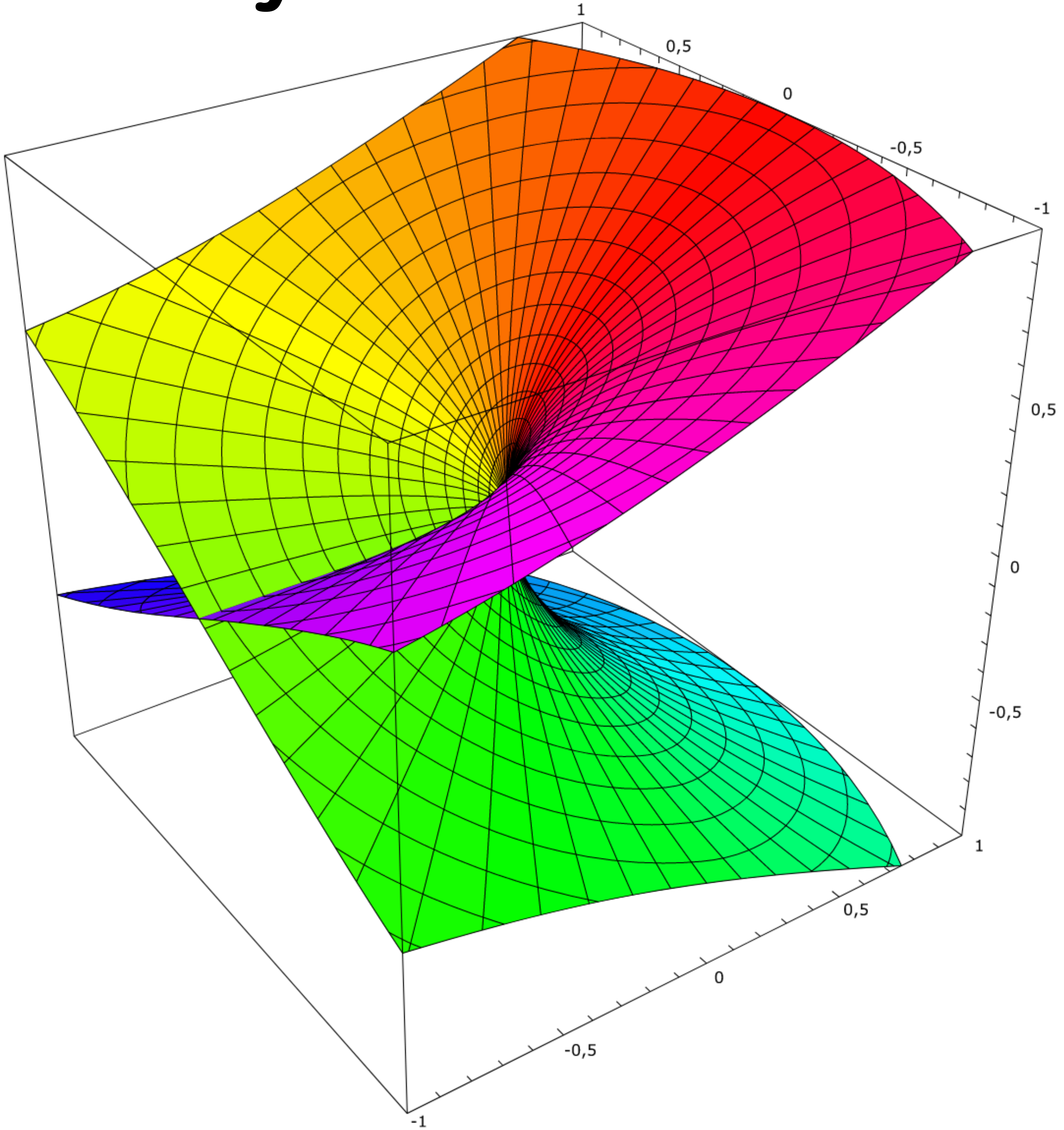
Puiseux, 1850

$$y^2 = x^3 + x$$

$$\Rightarrow \quad y = x^{1/2}\left(1 + x^2/2 - x^4/8 + \cdots\right)$$

$\mapsto$ idea of *monodromy*, 200 years later.

Monodromy



$y = x^{3/2}$

New transcendental functions

"It is well known that the central problem of the whole of modern mathematics is the study of the transcendental functions defined by differential equations."

Klein, Lecture I, Evanston Colloquium, 1893.

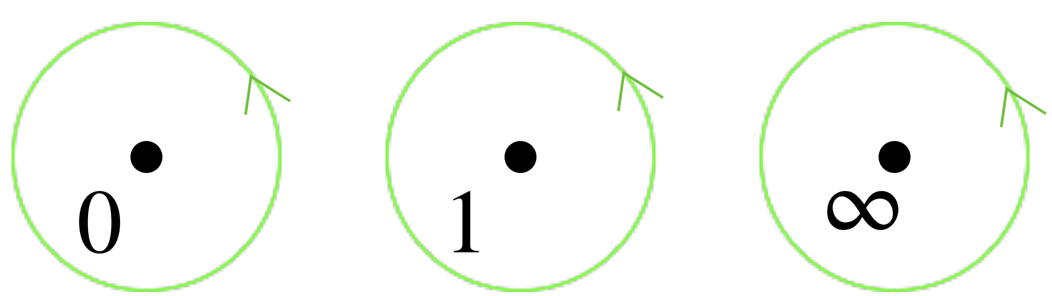
Prototypical example

The hypergeometric differential equation

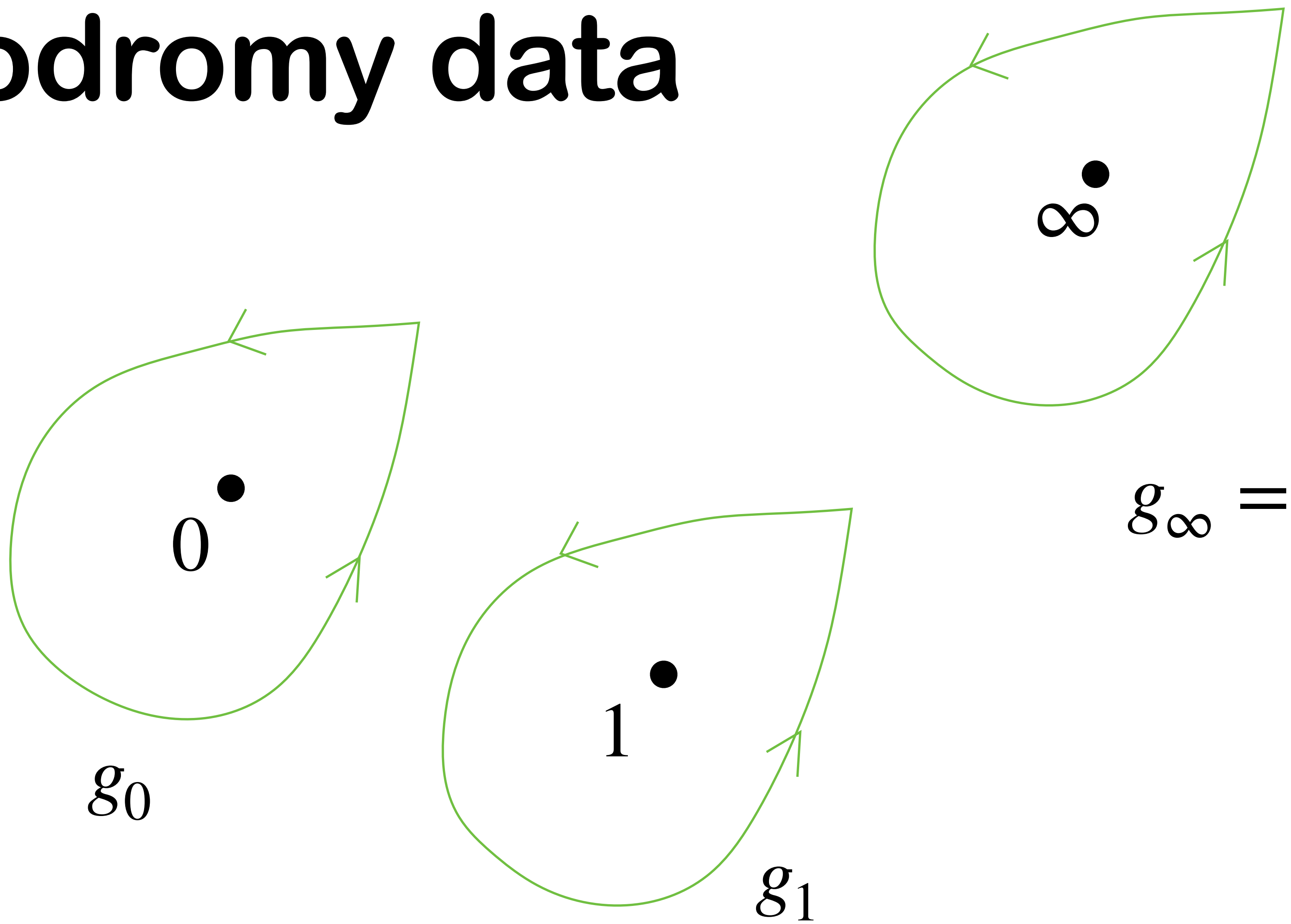
$$z(1-z)w_{zz} + \left(c - (a+b+1)z\right) w_z - abw = 0$$

has Fuchsian singularities (regular singularities) at $0, 1, \infty$.

The fundamental 2x2 matrix of solutions $Y(z)$ changes when analytically continued on a path around each singularity.



Monodromy data



$$g_\infty = \left(g_0 \circ g_1\right)^{-1}$$

g_0, g_1 are 2x2 matrices, which generate the monodromy group.

Data that remains invariant under simultaneous conjugation of such matrices is called *monodromy data.*

Lazarus Fuchs

1833 – 1902

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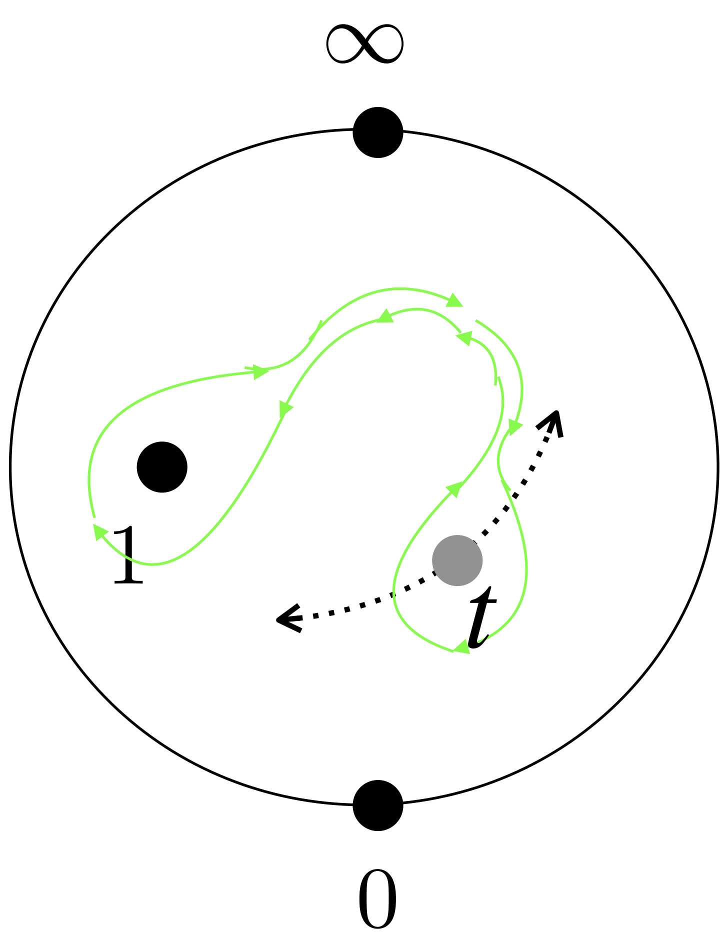
Lazarus Fuchs

1833 – 1902

Richard Fuchs

1873 – 1944

Add another singularity



R. Fuchs 1905

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) Y$$

Find the condition under which *monodromy* data of this system stays invariant under deformation of t .

$\rightarrowtail$ *isomonodromy problem*

Fuchs equation

$$w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)(w')^2$$

$$- \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right) w'$$

$$+ \quad \frac{w(w-1)(w-t)}{t^2(t-1)^2}\left(\alpha - \frac{\beta t}{w^2}\right.$$

$$\left. + \quad \frac{\gamma(t-1)}{(w-1)^2} + \delta\,\frac{t(t-1)}{(w-t)^2}\right)$$

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A limiting form

- Take

$$t \mapsto 1 + \epsilon\, t$$

$$\delta \mapsto \frac{\delta}{\epsilon^2}$$

$$\gamma \mapsto \frac{\gamma}{\epsilon} - \frac{\delta}{\epsilon^2}$$

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$\epsilon \to 0$

$$\mathrm{P_V}: w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{t} + \frac{(w-1)^2}{t^2}\left(\alpha\, w + \frac{\beta}{w}\right) + \frac{\gamma\, w}{t} + \frac{\delta\, w(w+1)}{w-1}$$

Successive limits

$$P_{IV}: \quad w'' = \frac{1}{2w}(w')^2 + \frac{3w^3}{2} + 4tw^2 + 2(t^2 - \alpha)w + \frac{\beta}{w}$$

$$P_{III}: \quad w'' = \frac{1}{w}(w')^2 - \frac{w'}{t} + \frac{1}{t}(\alpha w^2 + \beta) + \gamma\, w^3 + \frac{\delta}{w}$$

$$P_{II}: \quad w'' = 2w^3 + tw + \alpha$$

$$P_{I}: \quad w'' = 6w^2 + t$$

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- One way to describe them is through associated isomonodromy problems.

  Ablowitz & Segur 1977, Jimbo & Miwa 1981, Its & Novokshenov 1986, Kapaev 1988, Kitaev & Kapaev 1993, Deift & Zhou 1995, Fokas & Zhou 1992, Fokas et al 2006.

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- Boutroux (1913) showed that

$$\begin{aligned} w(t) &= t^{1/2} u(z) \\ z &= \frac{4}{5} t^{5/4} \end{aligned} \quad \Rightarrow \quad u_{zz} = 6\,u^2 + 1 - \frac{u_z}{z} + \frac{4u}{25z^2}$$

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- So as $z \to \infty$, we find $u \sim \wp(z - z_0; -2; 2E)$, where

$$u_z^2 = 4u^3 + u + 2E$$

Modulated elliptic functions

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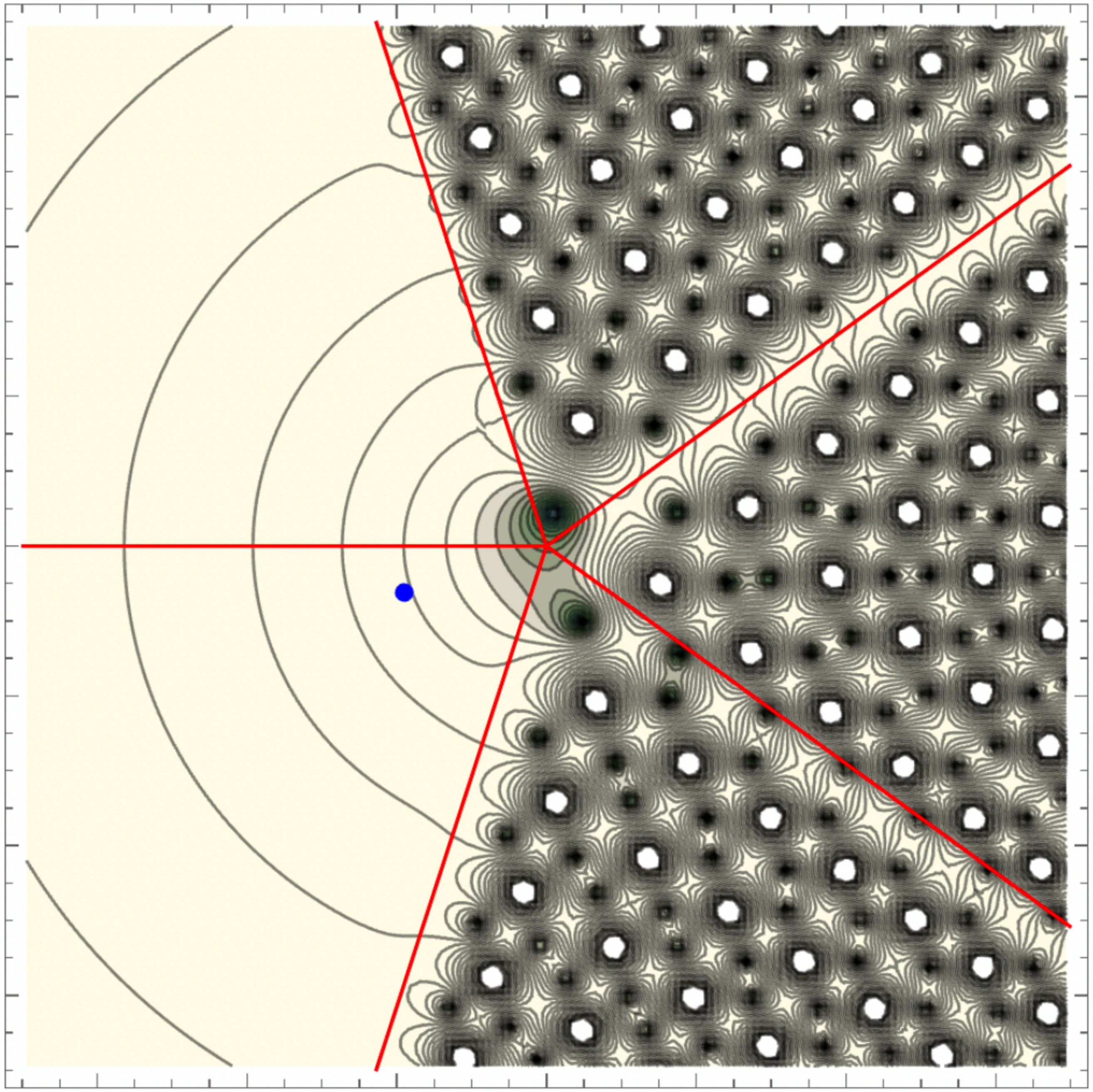
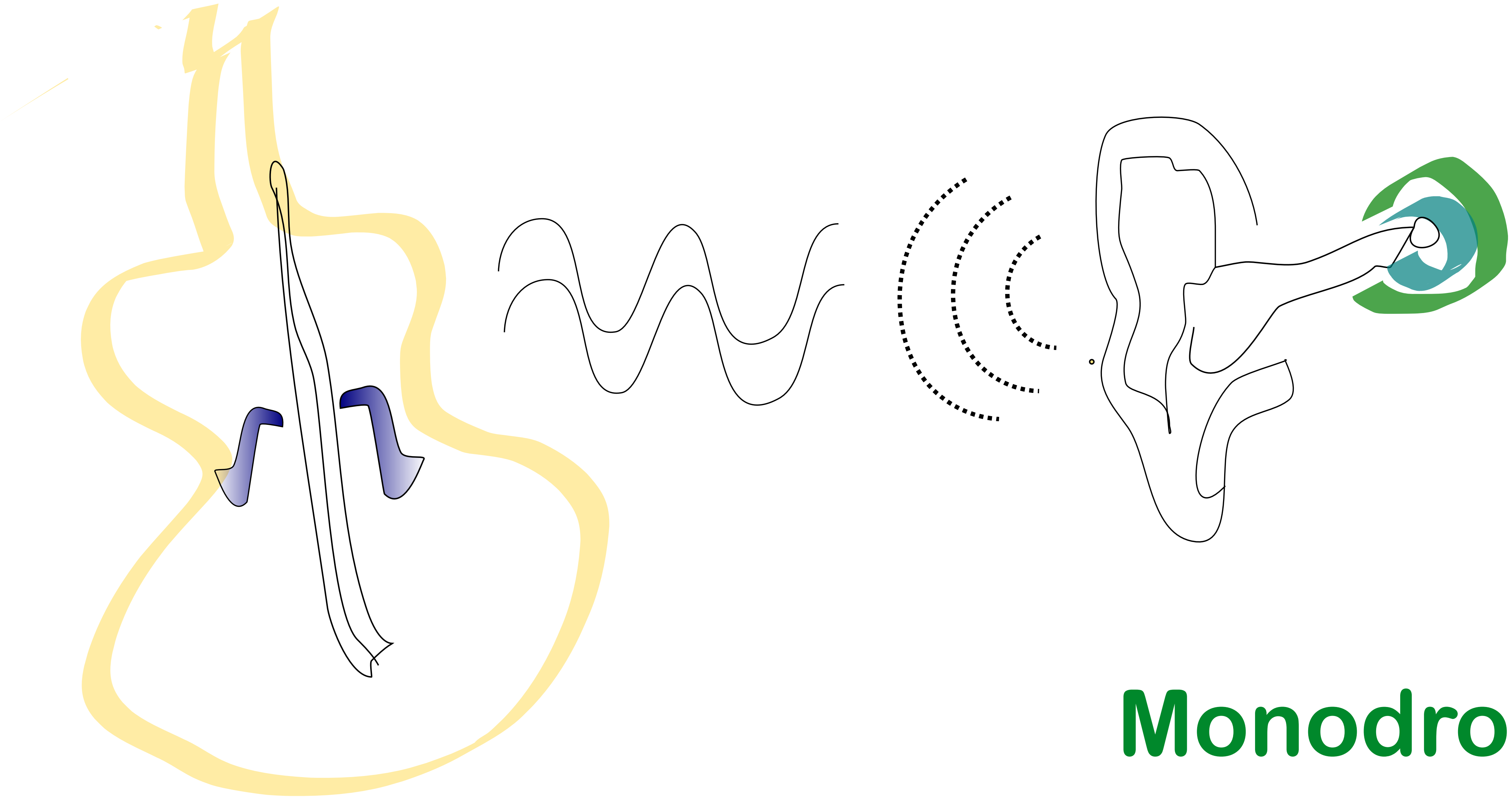
- The hyperbolic behaviours occur along boundaries of quadrants in z, which are sectors of width $2\pi/5$ in x.
- The algebraic behaviour occurs within two contiguous such sectors, and leads to *tronquée* solutions.



Poles of a tronqué solution of PI in t-plane from arXiv:2204.09062 Figure 1(b) by Alexander van Spaendonck and Marcel Vonk. (Figure is reflected from the original.)

How are such functions related to *isomonodromy* problems?



Monodromy

Isomonodromy problem for $\mathrm{P_I}$

- $\mathrm{P_I}$ is the compatibility condition for an associated 2x2 matrix linear system:

$$\frac{\partial Y}{\partial z} = A(z,t)Y \qquad A(z) = A_4 z^4 + A_2 z^2 + A_1 z + \frac{A_{-1}}{z}$$

Jimbo, Miwa 1981

$$\frac{\partial Y}{\partial t} = B(z,t)Y$$

- This system has an irregular singular point at $z = \infty$ and a regular singular point at $z = 0$.
- Monodromy now includes not only the information about how solutions change in the way described by Fuchs around 0, but also Stokes phenomena around ∞.

Stokes phenomena

$$Y \sim \big(I + O(1/z)\big)\exp\left(\left(\frac{4}{5}z^5 + t\,z\right)\sigma_3\right),\ z \to \infty$$

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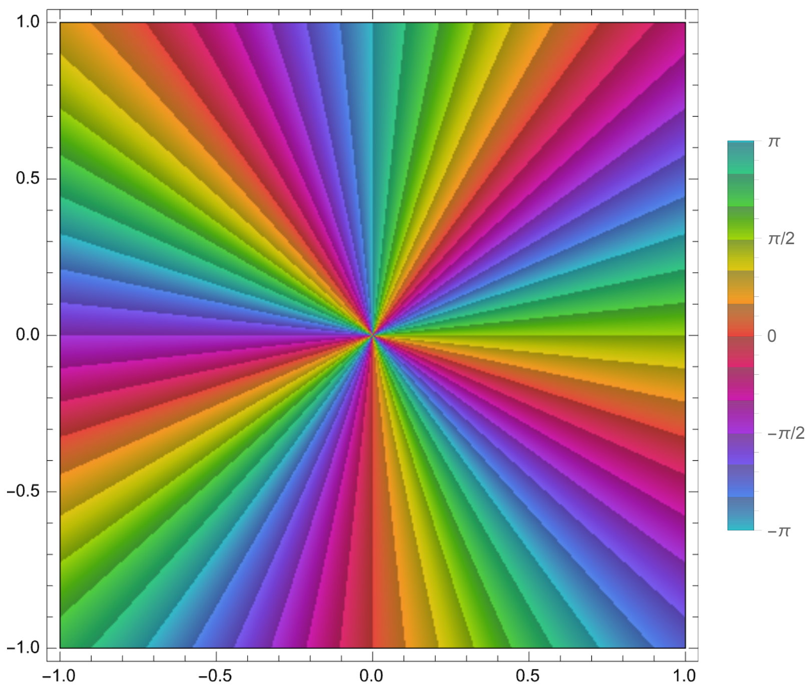
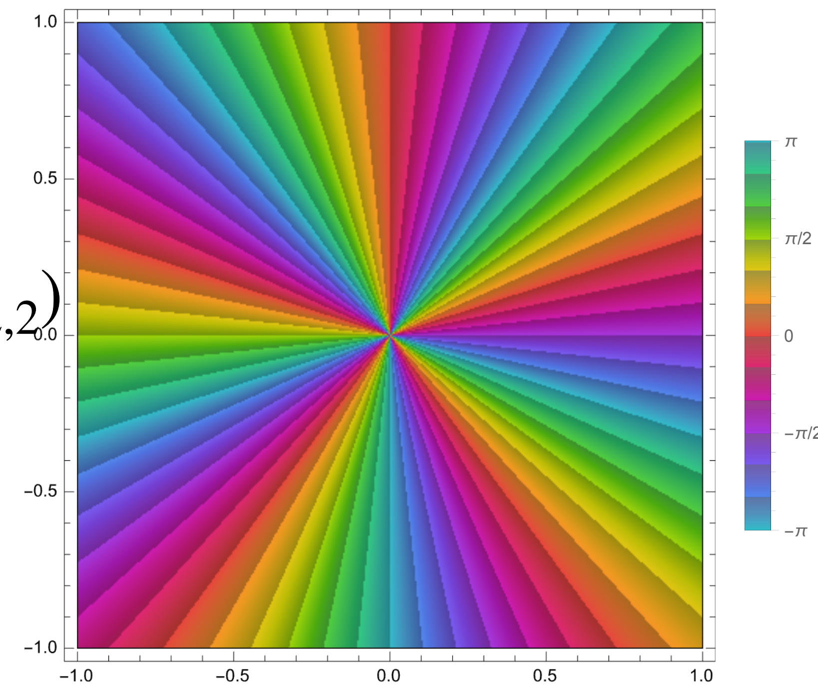


$\arg(Y_{2,2})$



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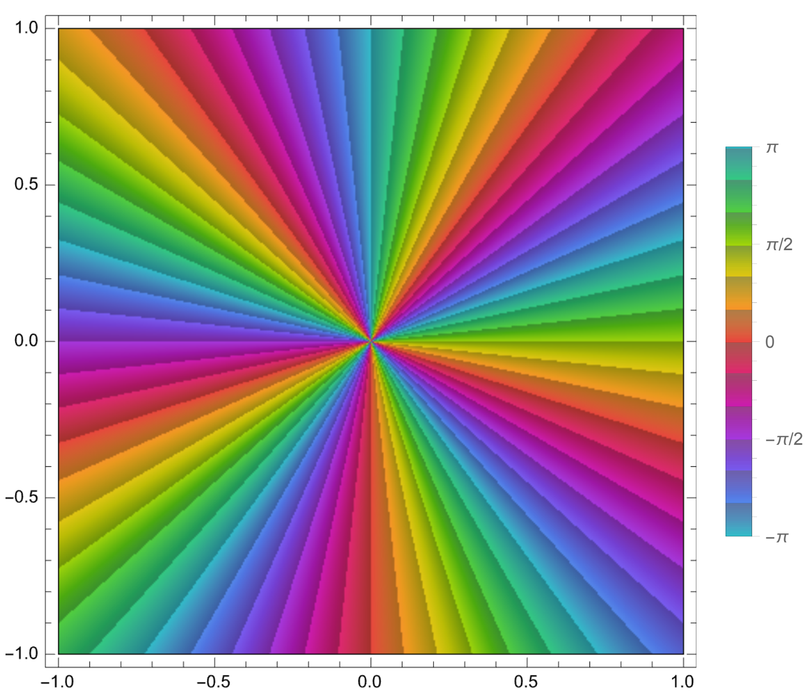

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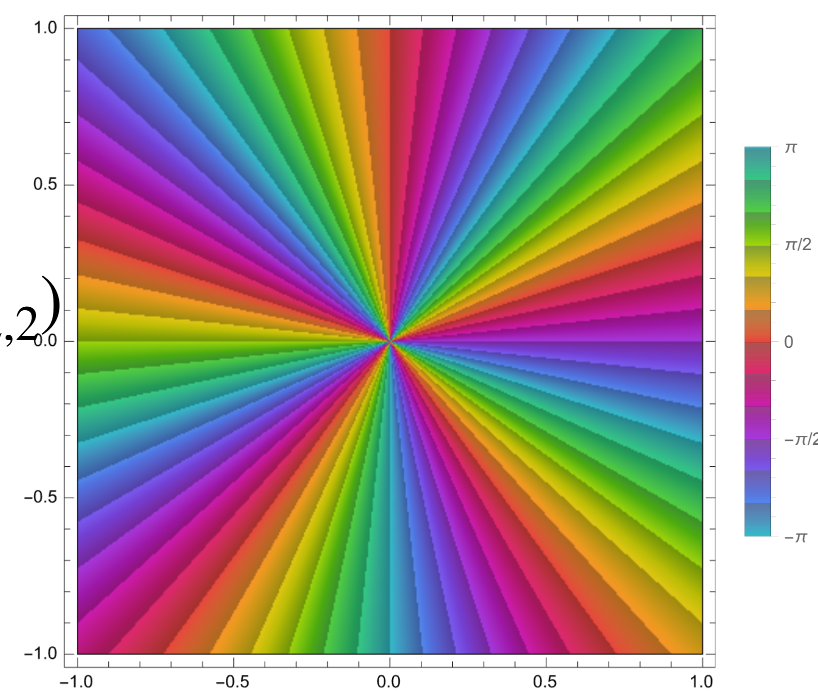


Choose a solution Y_k in a sector Ω_k. It must be related to the corresponding solution in a neighbouring sector by multiplication by Stokes matrices:

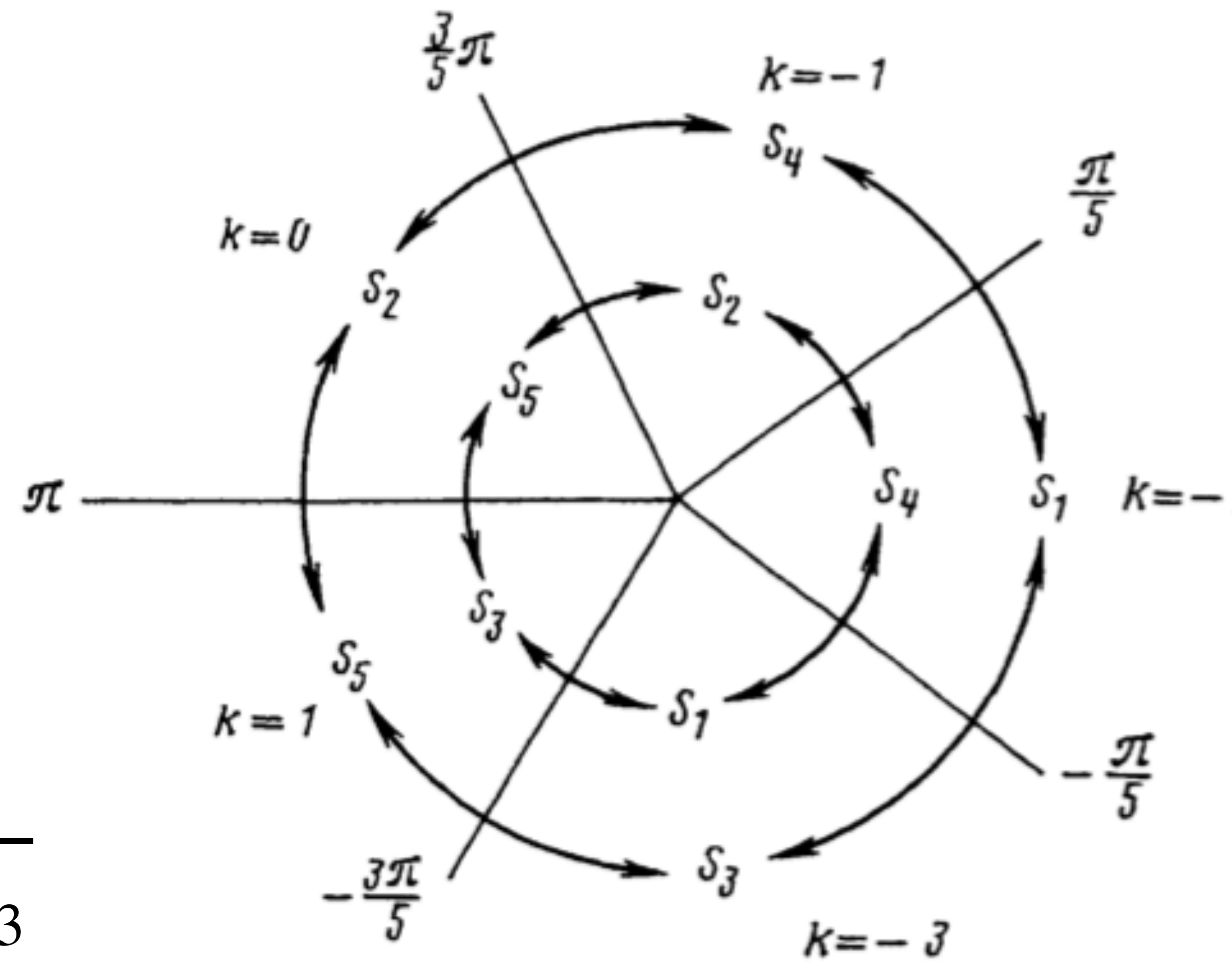
$$Y_{k+1} = Y_k S_k \qquad S_{2l} = \begin{pmatrix} 1 & 0 \\ s_{2l} & 1 \end{pmatrix}, \quad S_{2l+1} = \begin{pmatrix} 1 & s_{2l+1} \\ 0 & 1 \end{pmatrix}$$

Connections

- Connection to the solutions defined around $z = 0$.
- Symmetry condition (related to $\mathrm{P_I}$).
- These lead to interrelationships between parameters$\{s_k\}$, yielding

$$s_{k+5} = s_k \qquad s_4 = \frac{i - s_2}{1 + s_2 s_3}$$

$$s_1 = \frac{i - s_3}{1 + s_2 s_3} \qquad s_5 = i\left(1 + s_2 s_3\right)$$



Kapaev, 1988

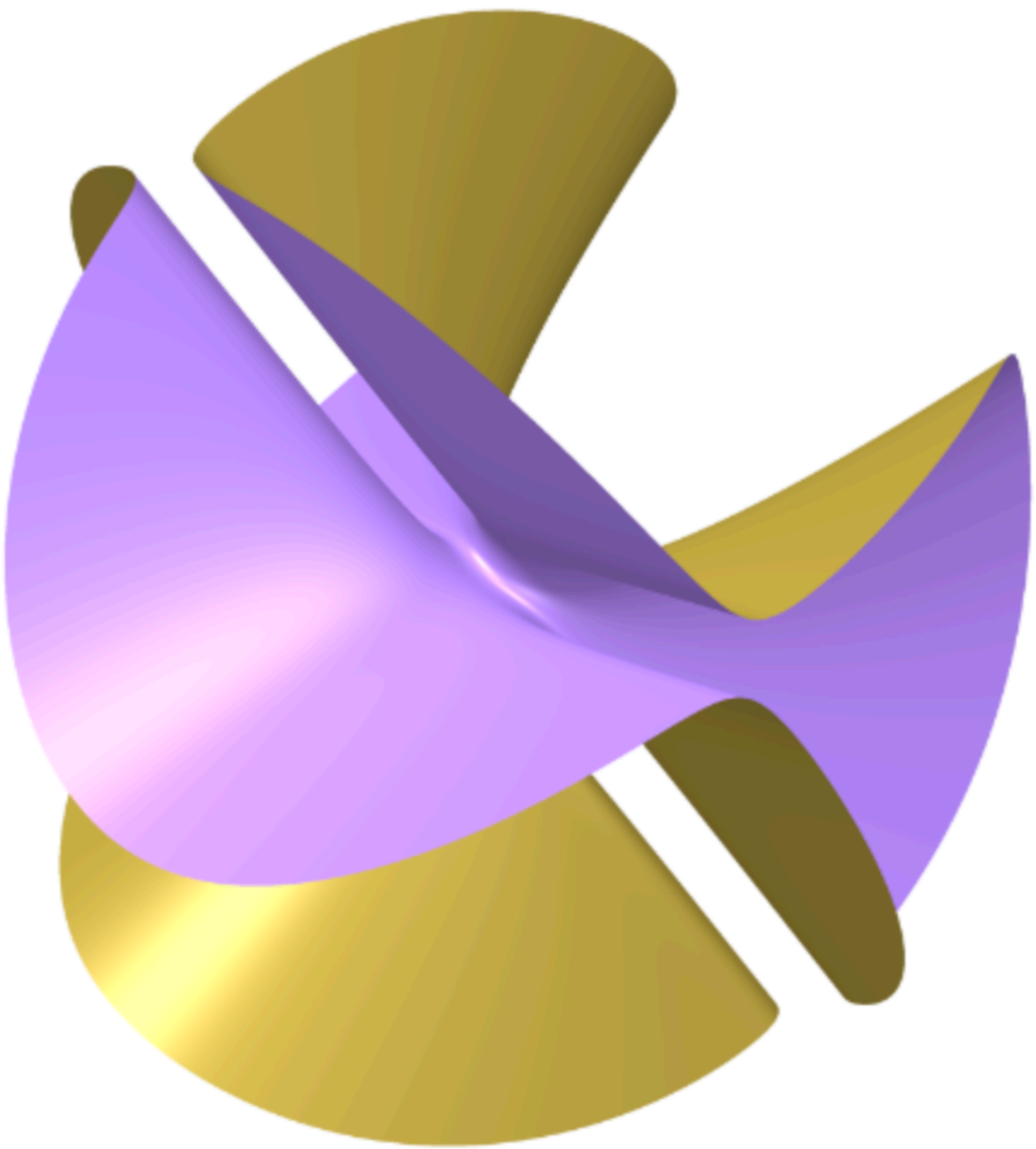
Kapev &Kitaev, 1993

Cubic surfaces

The relations boil down to

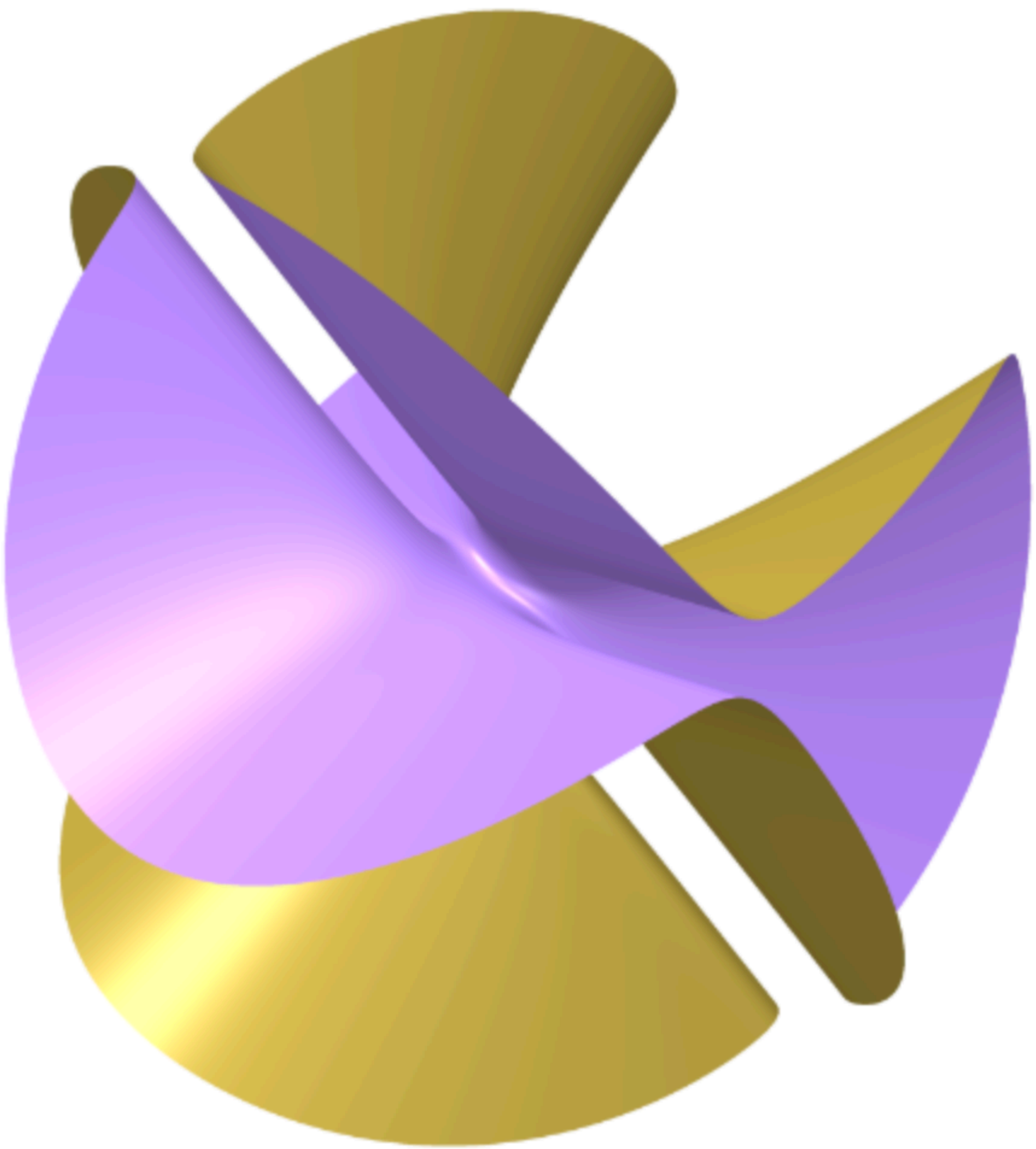
$$s_1 s_2 s_3 + s_1 + s_3 = i$$

That is, a cubic surface.



The cubic surface of P_I after scaling $s_1 = ix, s_2 = -iy, s_3 = iz.$

Cubic surfaces



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Similar cubic surfaces arise for the remaining Painlevé Equations, e.g., for P_{IV}

$$s_1 s_2 s_3 + s_1^2 + a s_1 - b s_2 + c s_3 + d = 0$$

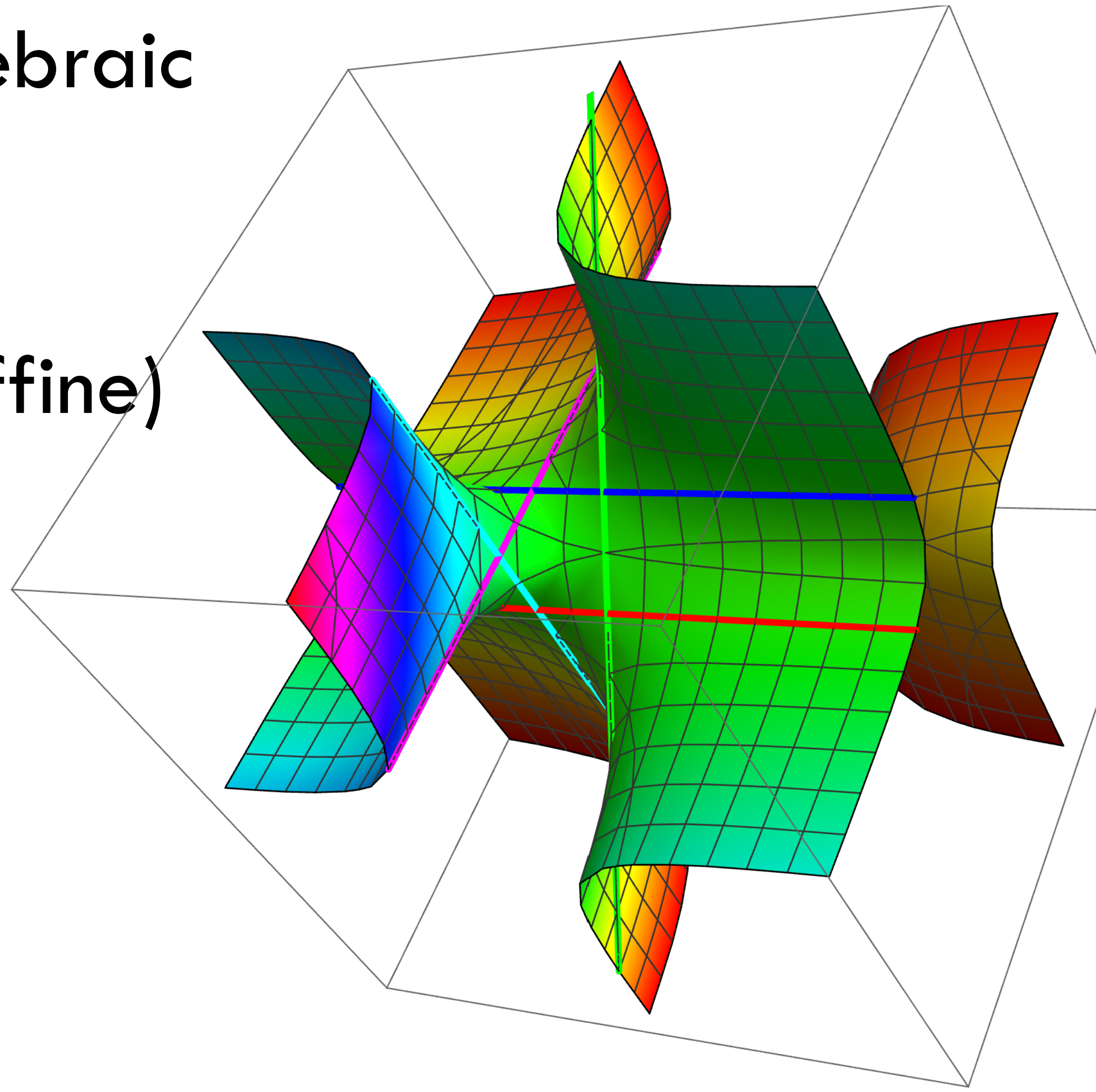
where a, b, c, d are related to the two parameters of P_{IV}.

Cubic surfaces

Cubic surfaces are celebrated in algebraic geometry, and lines on them play an important role.

The cubic surface for $\mathrm{P_I}$ contains 5 (affine) lines.

$$L_1 : \{s_1 = 0,\ s_3 = i\}$$
$$L_2 : \{s_1 = i,\ s_3 = 0\}$$
$$L_3 : \{s_2 = 0,\ s_1 + s_3 = i\}$$
$$L_4 : \{s_2 = i,\ s_3 = i\}$$
$$L_5 : \{s_1 = i,\ s_2 = i\}$$

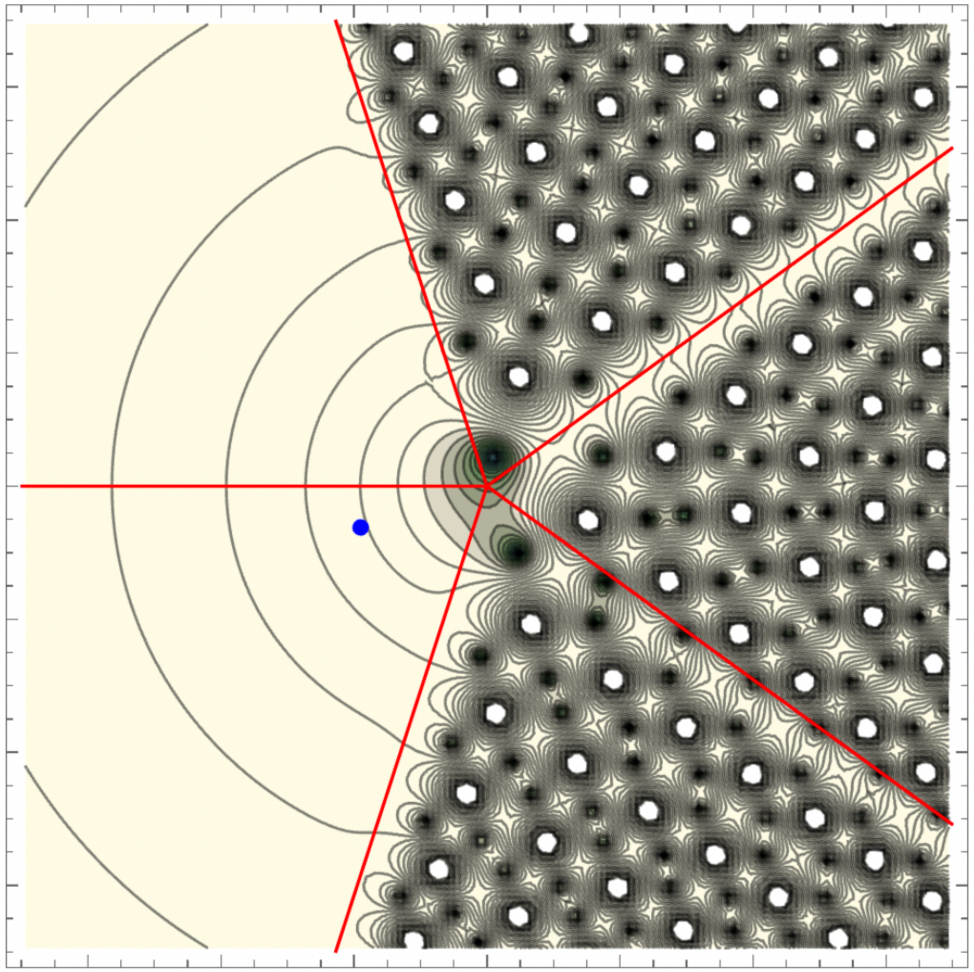


Each line L_j is crucially tied to a famous one-parameter asymptotic behaviour admitted by $\mathrm{P_I}$.

Tronquée solutions

$$w(t) \sim \left(\frac{-t}{6}\right)^{1/2} \sum_{j=0}^{\infty} \frac{a_j}{t^{5j/2}}$$

$$|t| \to \infty,\ 3\pi/5 < \arg(t) < 7\pi/5$$



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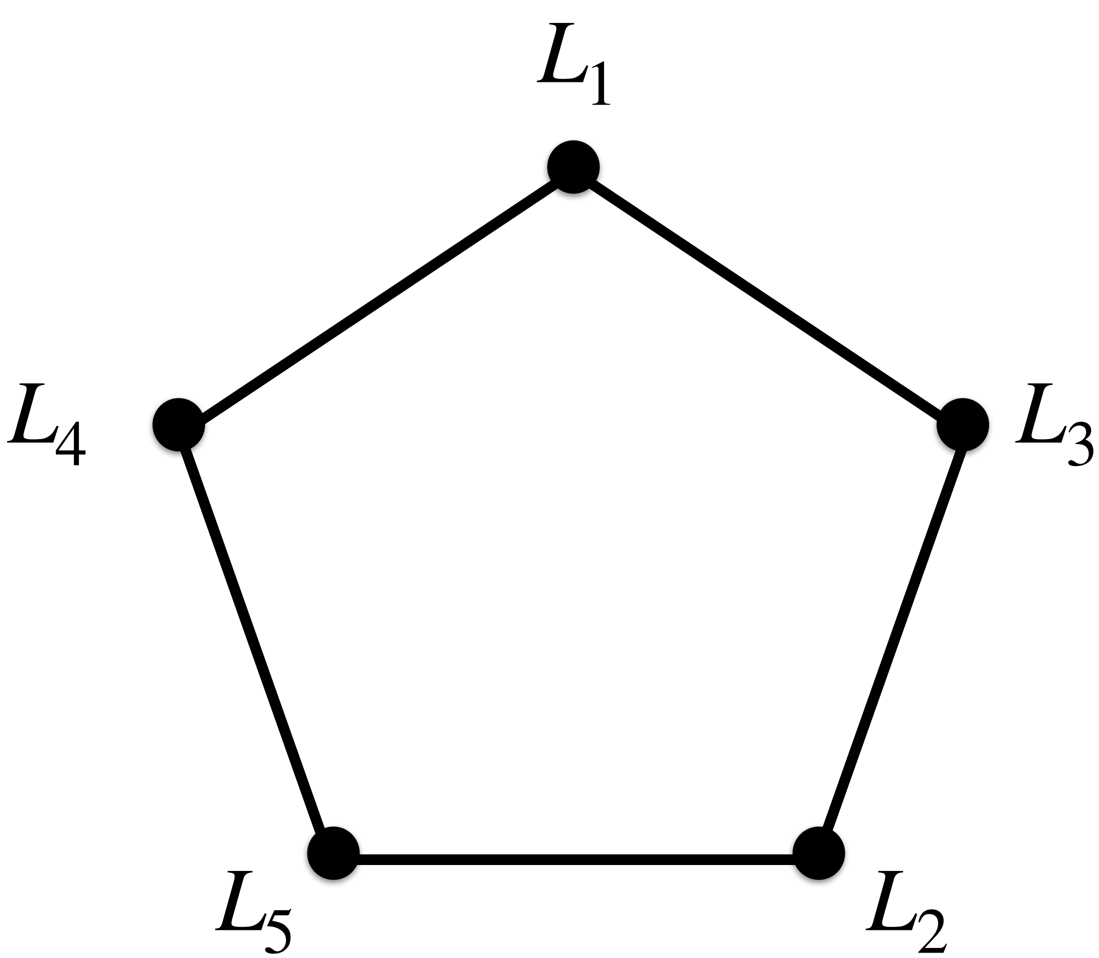
Tronquée solutions



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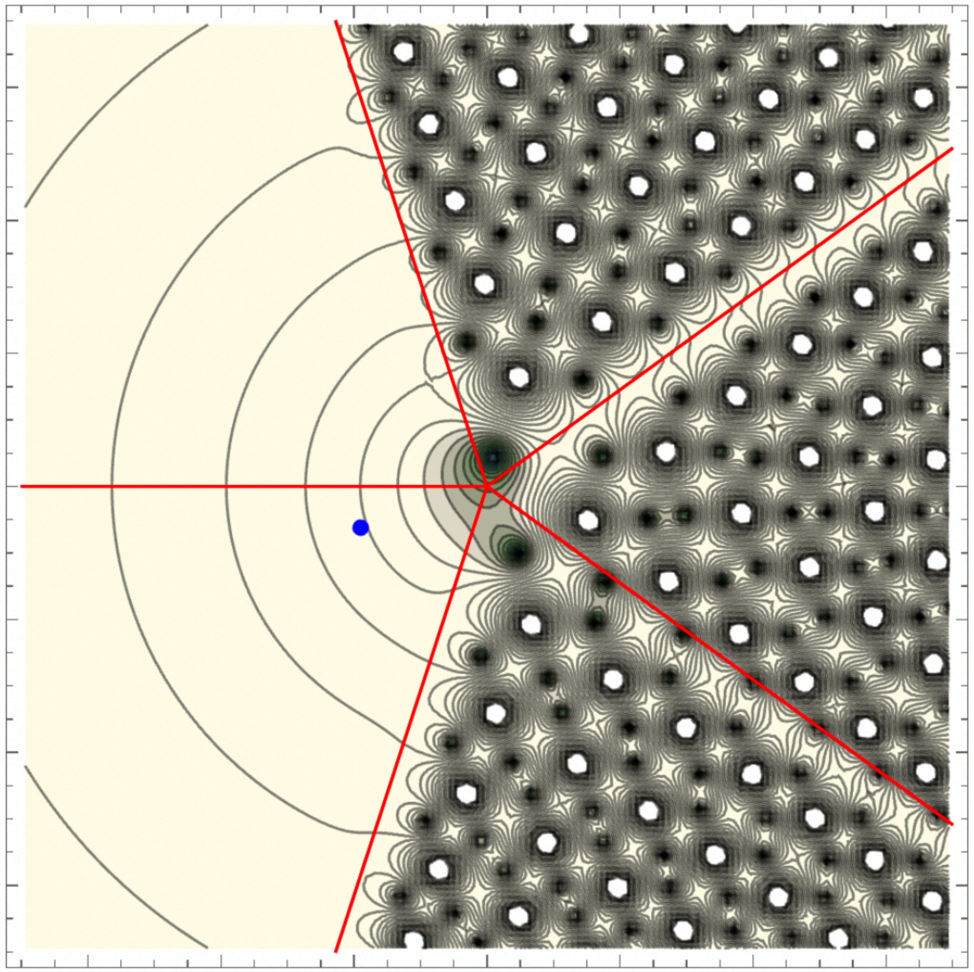

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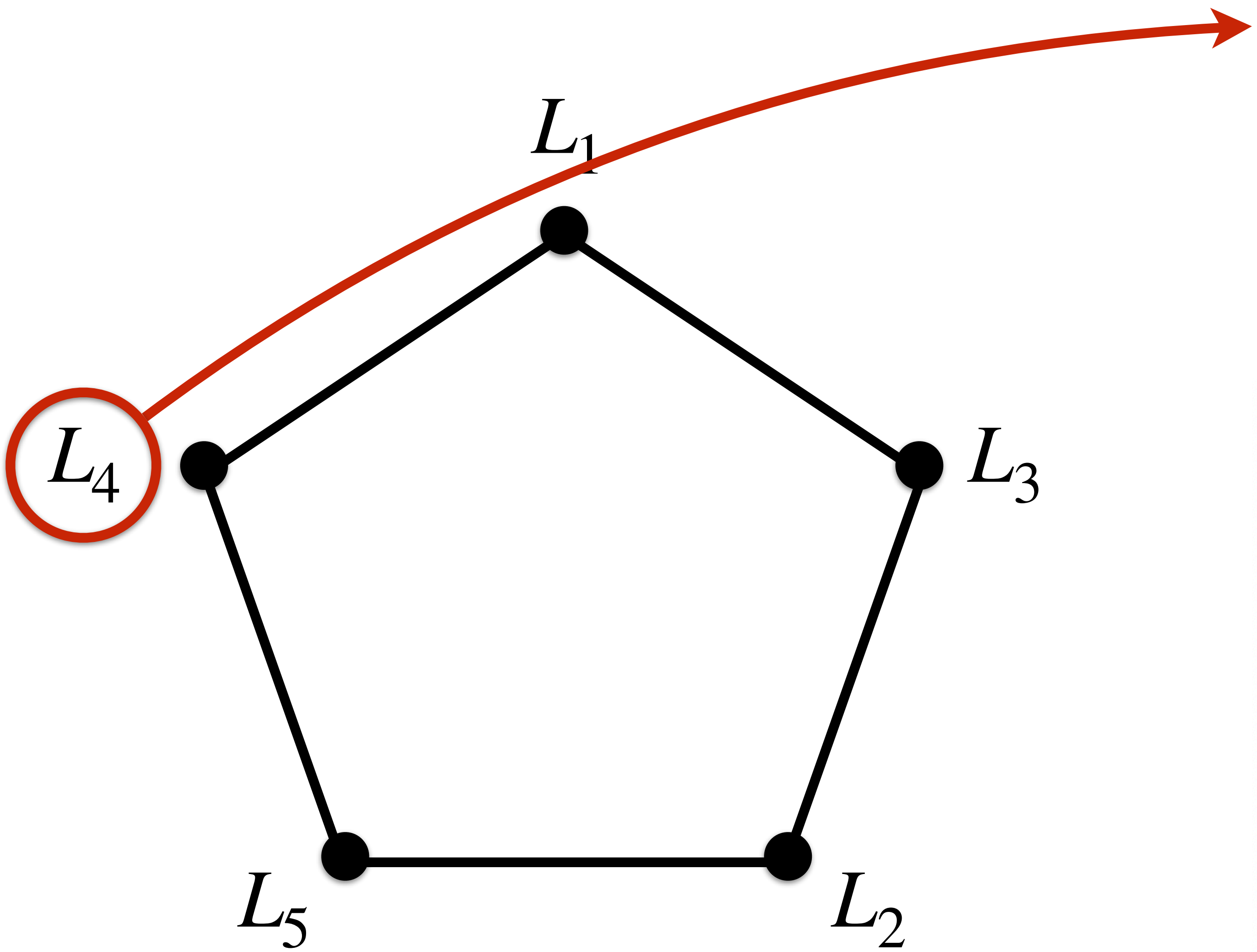

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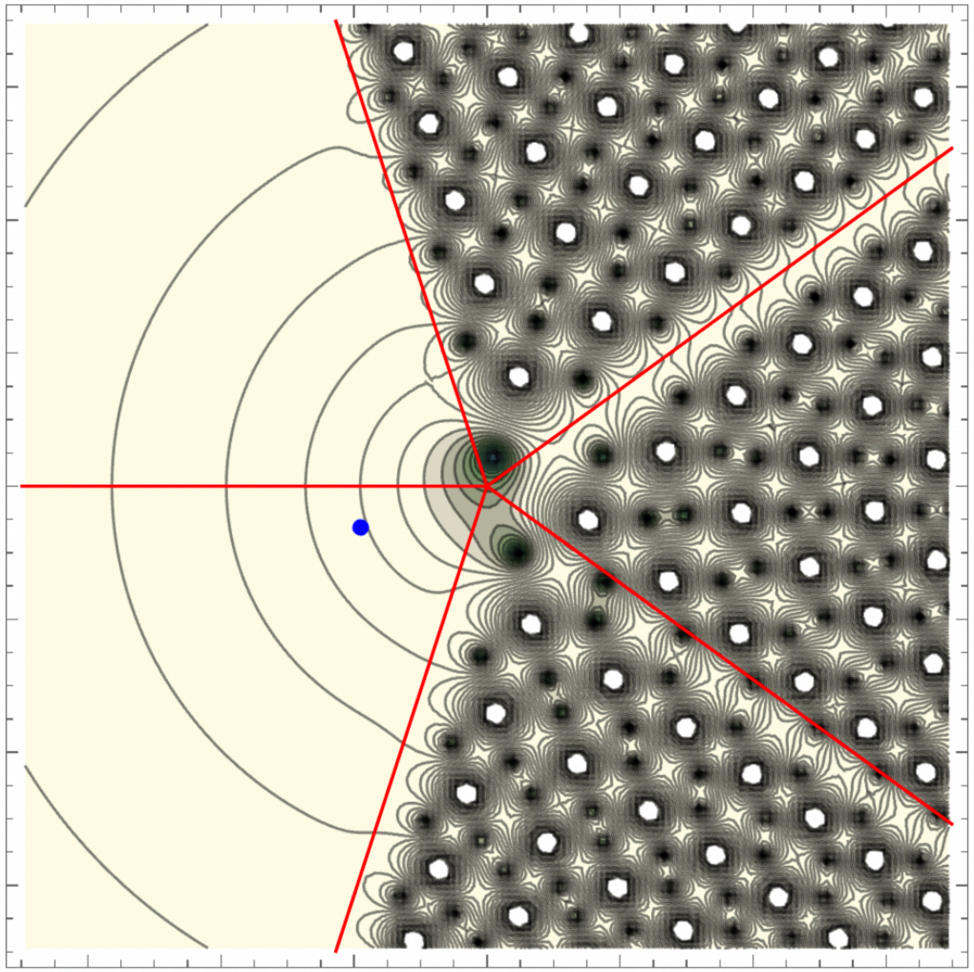
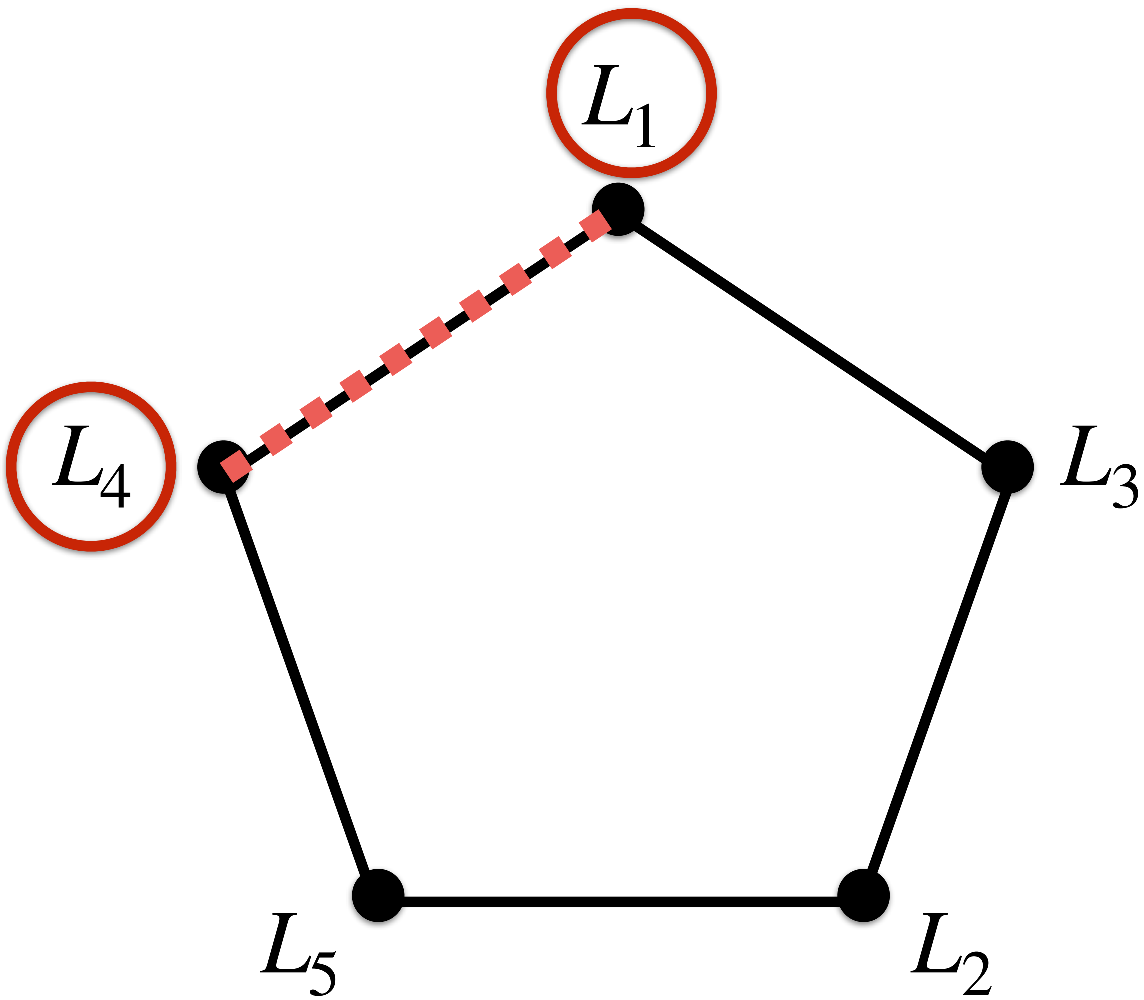
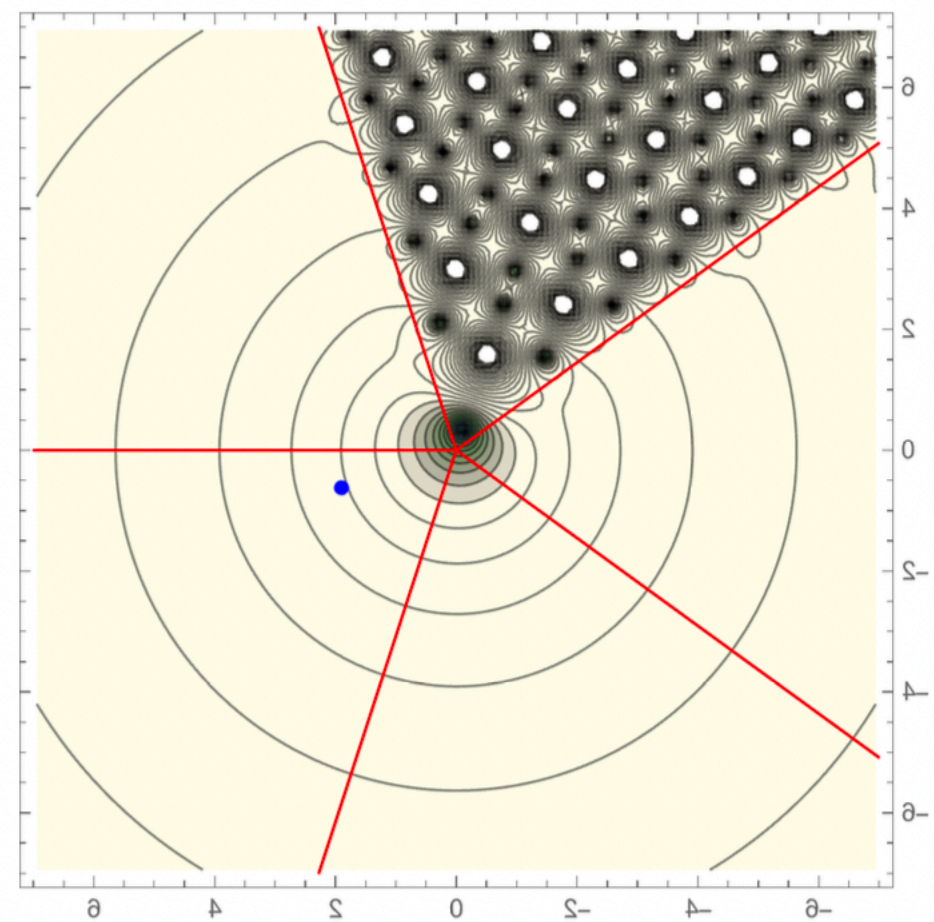


Poles of a tronqué solution of PI in t-plane from arXiv:2204.09062 Figure 1(b) by Alexander van Spaendonck and Marcel Vonk. (Figure is reflected.)

Tritronquée solutions



$$L_1 \cap L_4 : \{s_1 = 0,\ s_2 = i,\ s_3 = i\}$$



Poles of a tritronqué solution of PI in t-plane from arXiv:2204.09062 Figure 1(a) by Alexander van Spaendonck and Marcel Vonk. (Figure is reflected.)

Symmetric solutions

The monodromy surface

$$s_1 s_2 s_3 + s_1 + s_3 = i$$

contains points (ip, ip, ip), where

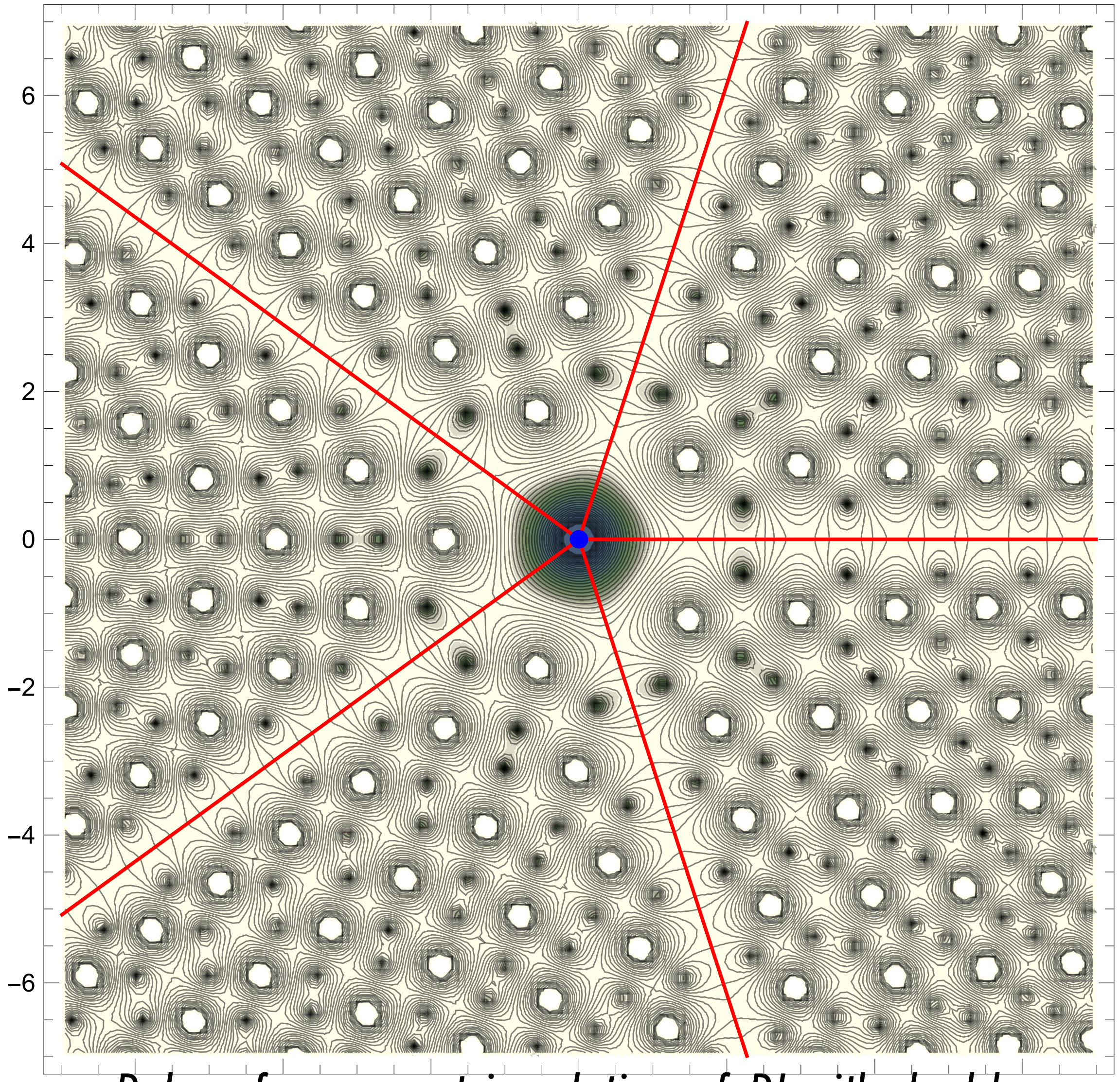
$$p^3 - 2p + 1 = 0$$

$$\Leftrightarrow \ (p-1)\left(p^2 + p + 1\right) = 0$$

Two of these points corresponds to symmetric solutions of P_I ($p = 1$ is *tritronquée*).

Kitaev, 1995

Symmetric solutions



Poles of a symmetric solution of PI with double zero at t=0 using code supplied by Marcel Vonk.

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What about discrete equations?

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Difference equations

$$F\left(\sigma^k(w(z)), \ldots, \sigma(w(z)), w(z), z\right) = 0$$

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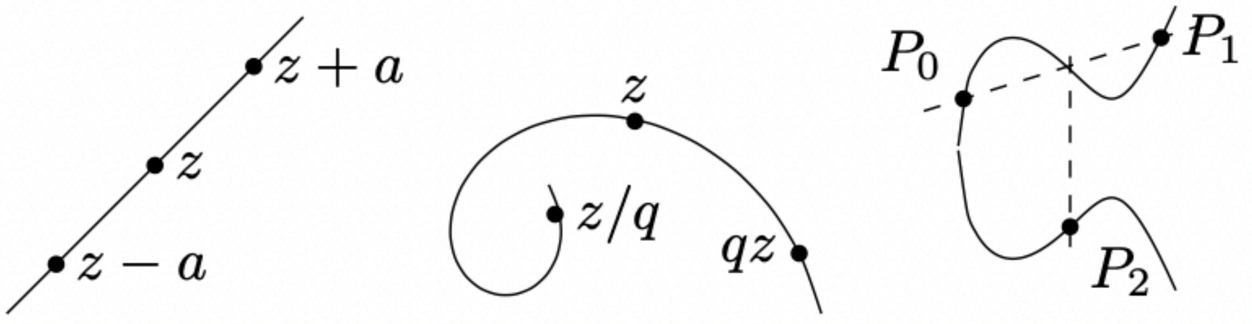

(a) σ (b) σ_q (c) σ_{ell}

$q \in \mathbb{C},\ 0 < |q| < 1$

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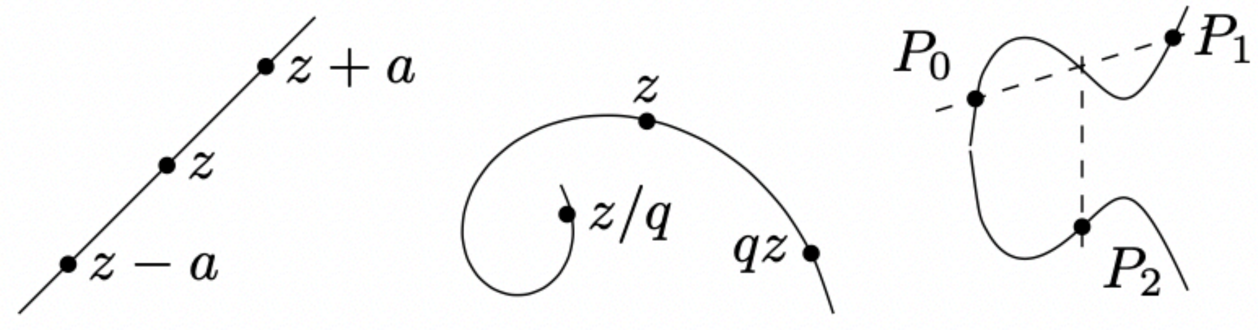

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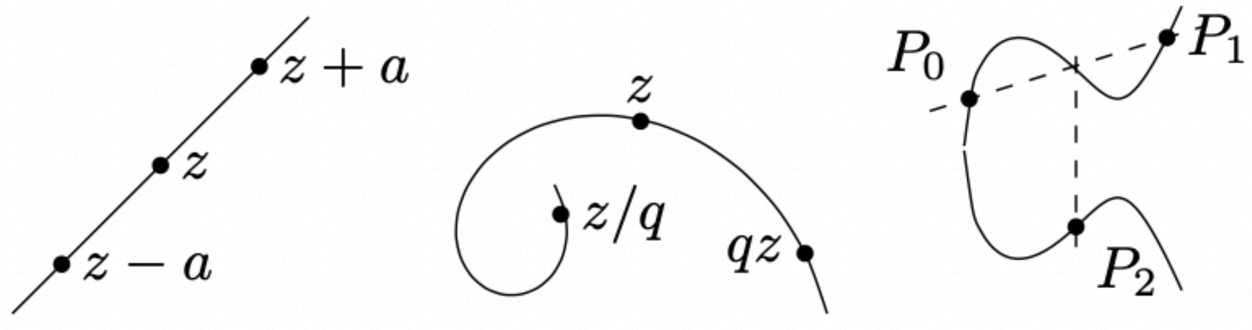

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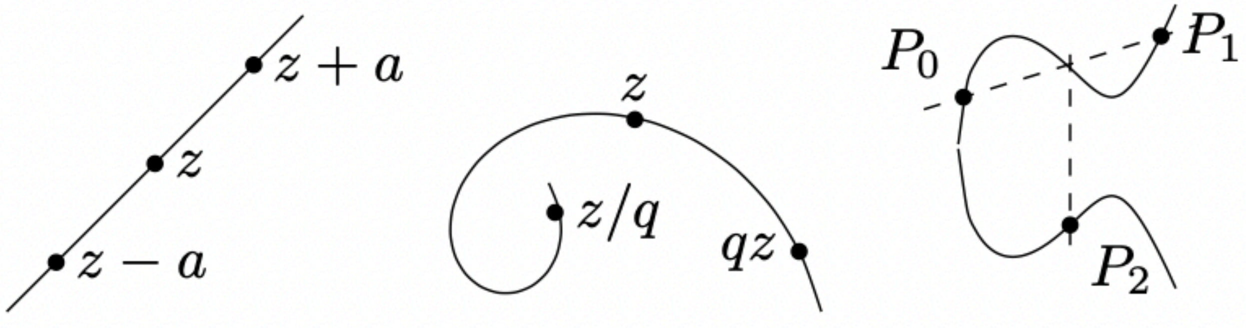

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additive -type	multiplicative -type	elliptic -type

Three types of Gamma functions

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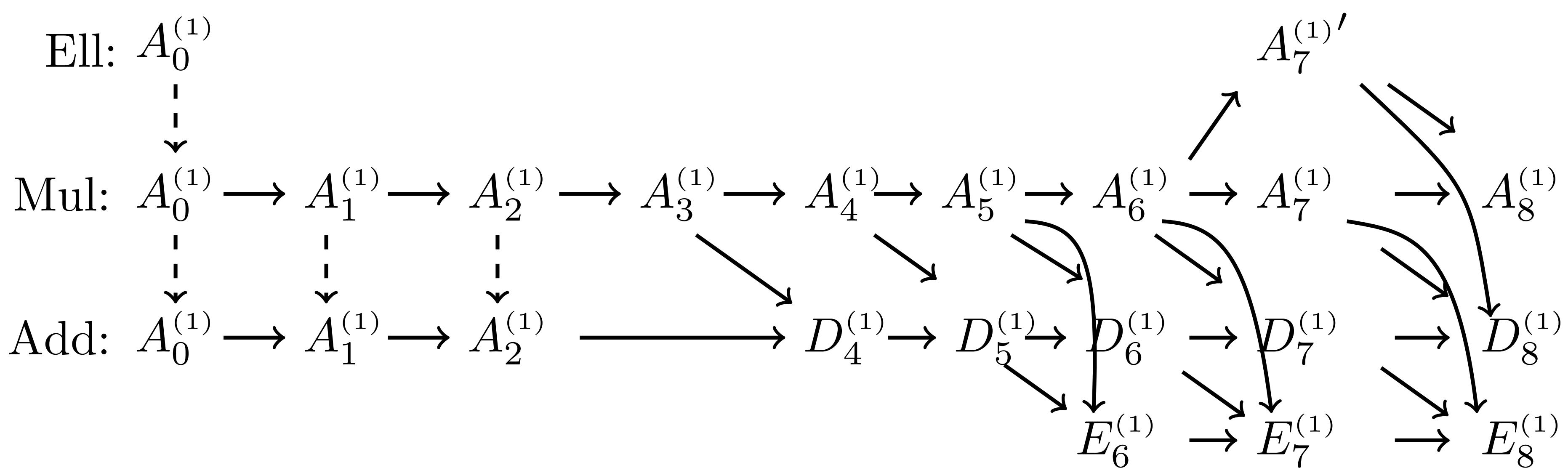
Related to Jacobi's theta function

Sakai's scheme

Initial value spaces R



Sakai 2001
Rains 2016

Sakai's scheme

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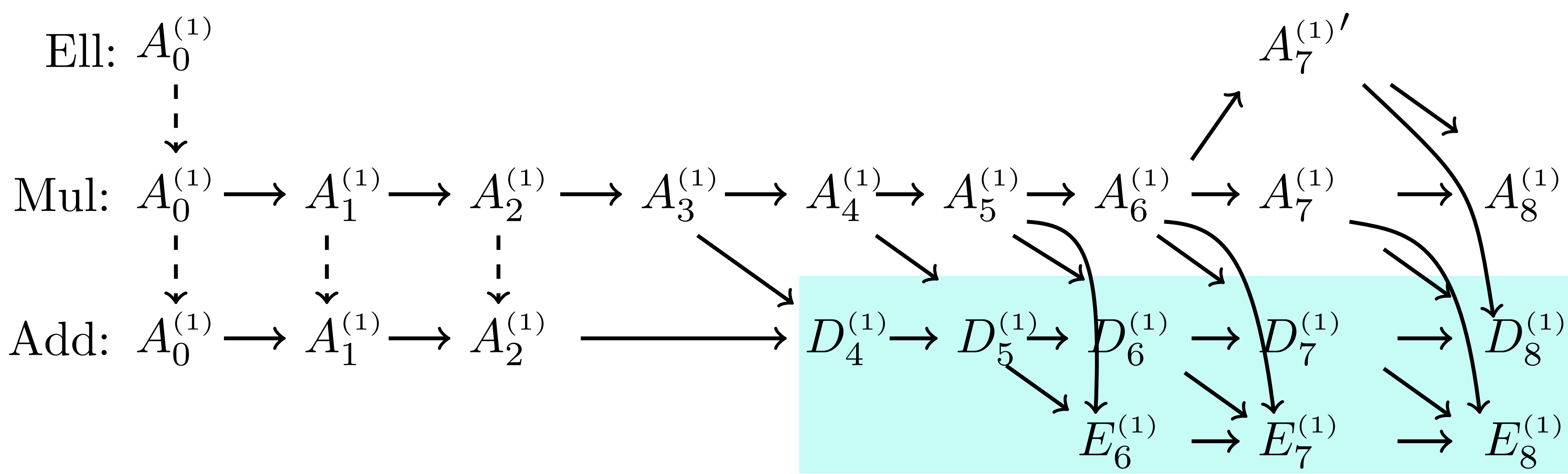

The Painlevé equations *Okamoto* 1979

Sakai 2001
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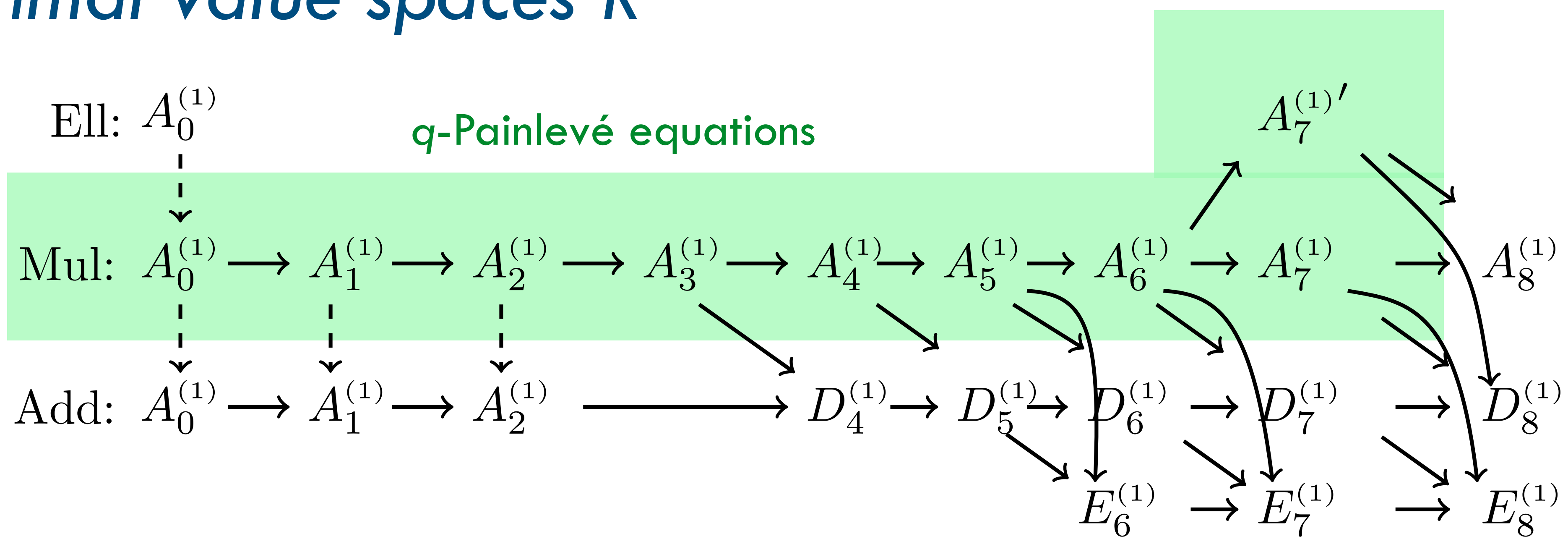

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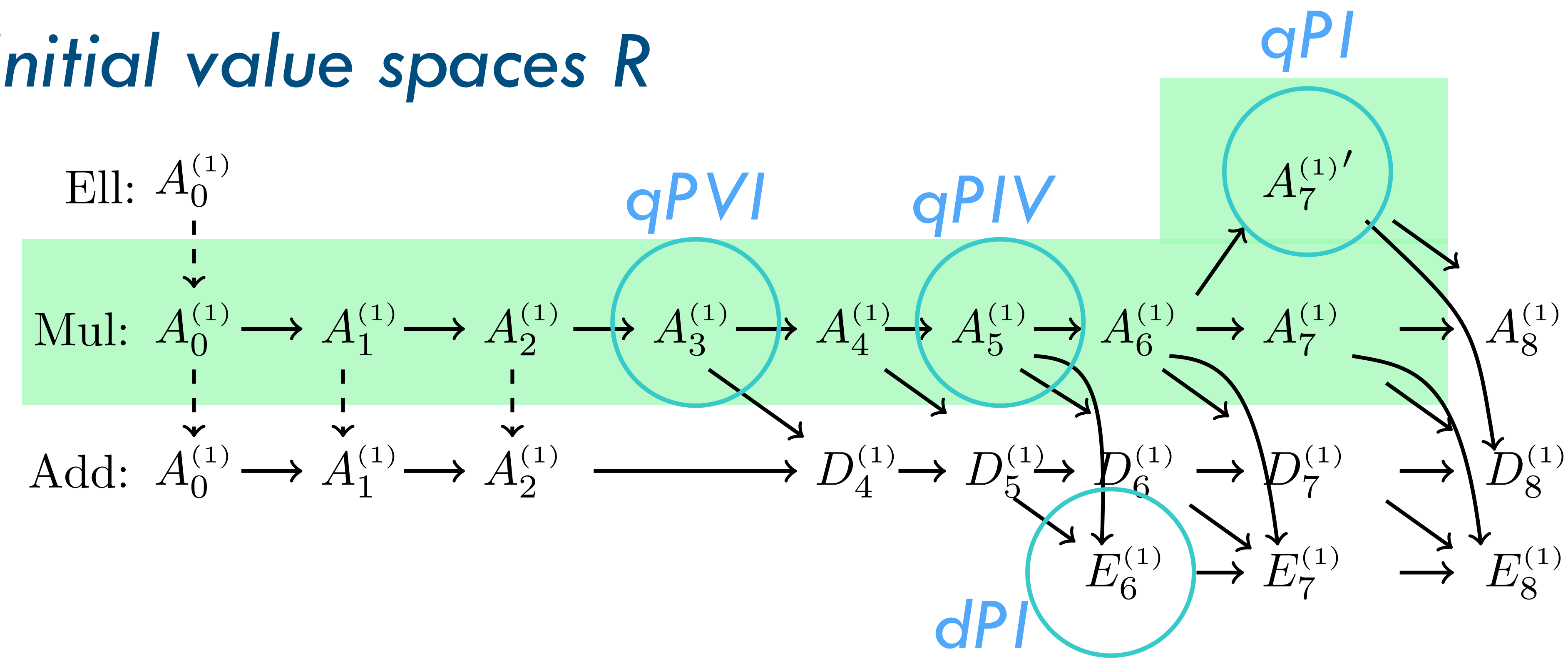

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dP$_l$

dP_I

An additive discrete equation arising from the differential equation P_{IV}

$$w_n'' = \frac{1}{2w_n}(w_n')^2 + \frac{3w_n^3}{2} + 4tw_n^2 + 2(t^2 - \alpha_n)w_n - \frac{\gamma_n^2}{2w_n}$$

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Bäcklund transformations of P_{IV}

$$\alpha_n = -\frac{n}{2} + c_0 + c_1(-1)^n, \quad \gamma_n = n - 2\,c_0 + \frac{2c_1}{3}(-1)^n$$

$$2w_n w_{n+1} = -\,w_n' - w_n^2 - 2tw_n + \gamma_n$$
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Adding these ⇒ the first discrete Painlevé equation or dP_I

$$w_n(w_{n+1} + w_n + w_{n-1}) = \gamma_n - 2tw_n$$

The case $c_0 = 0 = c_1$ is called the *string equation.*

dP_I : Asymptotic Analysis I

$$w_n = n^{1/2} u_n$$

$$\Rightarrow u_n(u_{n+1} + u_n + u_{n-1}) = 1 + \mathcal{O}(n^{-1/2}),\ n \to \infty$$

$K = (u_n u_{n-1} - 1)(u_{n-1} + u_n)$ is invariant to leading-order.

$$u_n \sim \frac{\left(3K + 6y_n\right)}{\left(6y_n - 2\right)}$$

gives $\quad y_n = \wp(n - n_0; 1/3; -2/(27) + K^2/4)$

What is the asymptotic behaviour of this solution as $n \to -\infty$?

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dP_I : Asymptotic Analysis II

There exist other behaviours

$u_{n\pm1} \sim u_n$

$$\Rightarrow u_n = \pm\sqrt{\frac{1}{3}} + \mathcal{O}(n^{-1/2}),\ n \to \infty$$

This expansion is divergent, hiding a free parameter – analogous to tronquée solutions

Late terms and Stokes phenomena were studied in J. & Lustri, 2015.

But, no one knows how solutions connect across Stokes sectors.

q-PI: Asymptotic analysis

$$\overline{w}\,\underline{w} = \frac{w-t}{w^2}, \qquad \overline{w} = w(qt),\ \underline{w} = w(t/q),\ 0 < |q| < 1$$

There exists a vanishing solution, $w(t)$ s.t.

$$w(t) \sim \sum_{n=1}^{\infty} b_n t^n, \text{ as } t \to 0$$

with late terms given by

$$b_{3\,p+1} = \mathcal{O}\Big(|q|^{-3\,p\,(p-1)/2} \prod_{k=0}^{p-1} \big(1+q^{3k}\big)^2\Big).$$

J. 2014

$$b_{3\,p+2} = 0,\ \ b_{3\,p+3} = 0,\ \ \forall p \geq 0$$

A factorially divergent series, hiding a free parameter.

There also exists a periodic behaviour, with period 3.

Towards global information

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- Recent developments of a q-version of Riemann-Hilbert theory give detailed information about monodromy surfaces (*NJ & Pieter Roffelsen, 2021, 2022. See also Ohyama, Ramis & Sauloy, 2020*) for certain cases.
- Geometric information on these surfaces give connections for "*tronquée*"-like solutions.

$q\mathrm{P}_{\mathrm{IV}}$

$qP(A_5^{(1)})$

$$q\mathrm{P}_{\mathrm{IV}}: \begin{cases} \dfrac{\overline{f}_0}{a_0 a_1 f_1} = \dfrac{1 + a_2 f_2 (1 + a_0 f_0)}{1 + a_0 f_0 (1 + a_1 f_1)}, \\ \dfrac{\overline{f}_1}{a_1 a_2 f_2} = \dfrac{1 + a_0 f_0 (1 + a_1 f_1)}{1 + a_1 f_1 (1 + a_2 f_2)}, \\ \dfrac{\overline{f}_2}{a_2 a_0 f_0} = \dfrac{1 + a_1 f_1 (1 + a_2 f_2)}{1 + a_2 f_2 (1 + a_0 f_0)}, \end{cases}$$

$$\overline{f}_j = f_j(qt)$$

$$f_0\, f_1\, f_2 = t^2, \qquad a_0\, a_1\, a_2 = q$$

Kajiwara, Noumi, Yamada 2001

$q\mathrm{P_{IV}}$-Linear problem

- The corresponding linear problem for $q\mathrm{P_{IV}}$ is

$$Y(qz,t) = A(z,t)\, Y(z,t)$$

$$A(z,t) = A_0(t) + A_1(t)\, z + A_2(t)\, z^2 + A_3(t)\, z^3$$

A_0 has eigenvalues $\pm i$

$$A_3 = q\, a_0^2\, a_2\, i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$$

$$|A(z)| = (1 - a_0\, z)(1 + a_0\, z)(1 - a_0 a_2 z)(1 + a_0 a_2 z)(1 - q\, z)(1 + q\, z)$$

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- These satisfy the conditions of Carmichael 1912, Birkhoff 1913 for existence, uniqueness of a solution and invertibility of the connection matrix.

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NJ and N. Nakazono, *PRSA* (2016)
arXiv:1503.04515

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qP_{IV}-Monodromy surface

$$
\begin{aligned}
&\theta_q(+a_0,+a_1,+a_2)\left(\theta_q(\lambda_0)p_1p_2p_3-\theta_q(-\lambda_0)\right)\\
&-\theta_q(-a_0,+a_1,-a_2)\left(\theta_q(\lambda_0)p_1-\theta_q(-\lambda_0)p_2p_3\right)\\
&+\theta_q(+a_0,-a_1,-a_2)\left(\theta_q(\lambda_0)p_2-\theta_q(-\lambda_0)p_1p_3\right)\\
&-\theta_q(-a_0,-a_1,+a_2)\left(\theta_q(\lambda_0)p_3-\theta_q(-\lambda_0)p_1p_2\right)=0
\end{aligned}
$$

A cubic surface

Note:

$$(\xi;q)_\infty = \prod_{k=0}^{\infty}(1-q^k\xi)$$

$$\theta_q(\xi) = (\xi;q)_\infty\,(q/\xi;q)_\infty$$

$$\theta_q(\xi_1,\ldots,\xi_n) = \theta_q(\xi_1)\ldots\theta_q(\xi_n)$$

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N. Joshi and P. Roffelsen, *Commun. Math. Phys* (2021) arXiv:1911.05854

$q\mathrm{P}_{\mathrm{VI}}$

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$$\begin{cases} f\bar{f} &= \dfrac{(\bar{g}-\kappa_0 t)(\bar{g}-\kappa_0^{-1}t)}{(\bar{g}-\kappa_\infty)(\bar{g}-q^{-1}\kappa_\infty^{-1})}, \\ g\bar{g} &= \dfrac{(f-\kappa_t t)(f-\kappa_t^{-1}t)}{q(f-\kappa_1)(f-\kappa_1^{-1})}, \end{cases}$$

$$q \in \mathbb{C},\ 0 < |q| < 1,\ \bar{f} = f(qt),\ \bar{g} = g(qt)$$

Jimbo, Sakai 1996

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Continuum limit of qP_{VI}

$$q \to 1,\ \kappa_j = q^{k_j},\ f \to u,\ g \to (u-t)/(u-1)$$

$$\alpha = \frac{(2k_\infty + 1)^2}{2}, \quad \beta = -2k_0^2, \quad \gamma = 2k_1^2, \quad \delta = \frac{1-4k_t^2}{2}$$

$$u_{tt} = \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)\frac{u_t^2}{2} - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u_t + \frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{u^2} + \frac{\gamma(t-1)}{(u-1)^2} + \frac{\delta t(t-1)}{(u-t)^2}\right)$$

P_{VI} the sixth Painlevé equation

$q\mathrm{P}_{\mathrm{VI}}$-monodromy surface

The q-monodromy surface for $q\mathrm{P}_{\mathrm{VI}}$ is given by two equations $T = 0$, where one is

$$T := T_{12}p_1p_2 + T_{13}p_1p_3 + T_{14}p_1p_4 + T_{23}p_2p_3 + T_{24}p_2p_4 + T_{34}p_3p_4$$

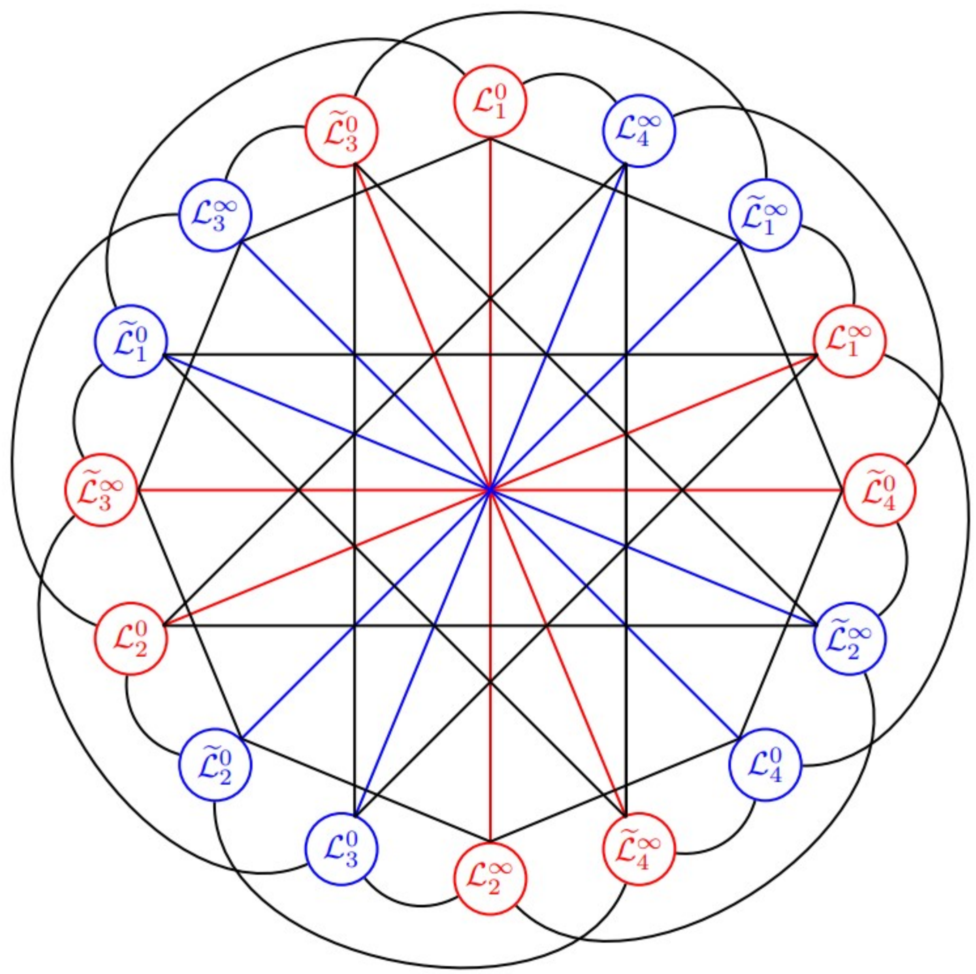
$$\begin{aligned}
T_{12} &= \theta_q\left(\kappa_t^2, \kappa_1^2\right)\theta_q\left(\kappa_0\kappa_\infty^{-1}t_0, \kappa_0^{-1}\kappa_\infty^{-1}t_0\right)\kappa_\infty^2,\\
T_{34} &= \theta_q\left(\kappa_t^2, \kappa_1^2\right)\theta_q\left(\kappa_0\kappa_\infty t_0, \kappa_0^{-1}\kappa_\infty t_0\right),\\
T_{13} &= -\theta_q\left(\kappa_t\kappa_1^{-1}t_0, \kappa_t^{-1}\kappa_1 t_0\right)\theta_q\left(\kappa_t\kappa_1\kappa_0^{-1}\kappa_\infty^{-1}, \kappa_0\kappa_t\kappa_1\kappa_\infty^{-1}\right)\kappa_\infty^2,\\
T_{24} &= -\theta_q\left(\kappa_t\kappa_1^{-1}t_0, \kappa_t^{-1}\kappa_1 t_0\right)\theta_q\left(\kappa_0\kappa_t\kappa_1\kappa_\infty, \kappa_t\kappa_1\kappa_\infty\kappa_0^{-1}\right),\\
T_{23} &= \theta_q\left(\kappa_t\kappa_1 t_0, \kappa_t^{-1}\kappa_1^{-1}t_0\right)\theta_q\left(\kappa_t\kappa_\infty\kappa_0^{-1}\kappa_1^{-1}, \kappa_0\kappa_t\kappa_\infty\kappa_1^{-1}\right)\kappa_1^2,\\
T_{14} &= \theta_q\left(\kappa_t\kappa_1 t_0, \kappa_t^{-1}\kappa_1^{-1}t_0\right)\theta_q\left(\kappa_1\kappa_\infty\kappa_0^{-1}\kappa_t^{-1}, \kappa_0\kappa_1\kappa_\infty\kappa_t^{-1}\right)\kappa_t^2.
\end{aligned}$$

This is a *Segre surface*.

N. J. and P. Roffelsen, On the monodromy manifold of *q*-Painlevé VI and its Riemann-Hilbert problem, arXiv:2202.10597

qP_{VI}-monodromy surface, ct'd

- The Segre surface contains 16 lines.



- Each line corresponds to a family of asymptotic behaviours at 0 and ∞.

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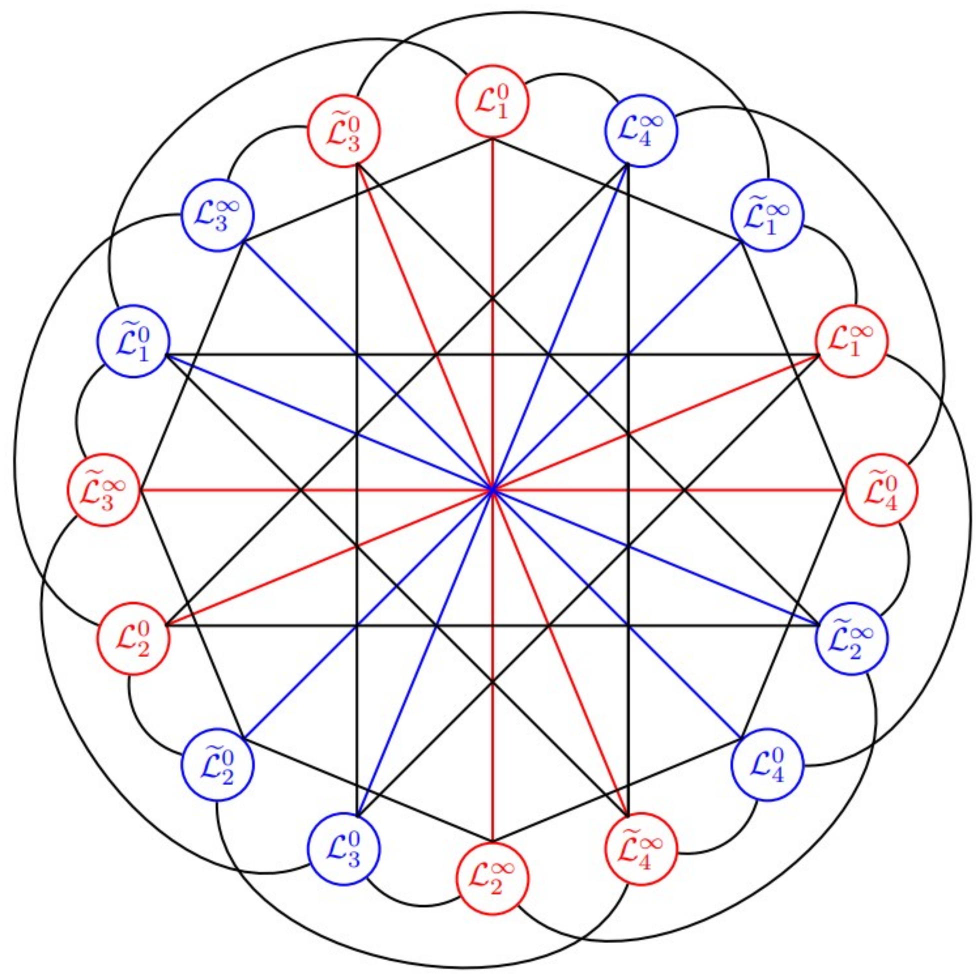

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N. Joshi and P. Roffelsen, *in preparation.*

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The journey is far from over…


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- Most additive discrete equations are related to differential equations, so, their solutions are seen as "understood". But their behaviours as functions of *n* remain unknown.
- What are the monodromy manifolds of other *q*-Painlevé equations?
- What are the behaviours of solutions of the elliptic-difference Painlevé equation?

References

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