On the onset of modulation instability in JONSWAP sea states

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NLS & Ocean waves

- Gravity waves can be represented by an **envelope**,

$$\eta(x,t) = \mathrm{Re}\left[u(x,t)e^{i(k_0\cdot x-\omega_0\cdot t)}\right], \quad \omega_0 = \sqrt{gk_0}.$$

- Asymptotically, the envelope satisfies the focusing cubic NLS

$$i\partial_t u + \frac{p}{2}\Delta u + \frac{q}{2}|u|^2 u = 0,$$

where $p = p(k_0) \gg 1$, $q = q(k_0) \ll 1$.
[Mei, *'Theory and applications of ocean surface waves'*, World Scientific]

Modulation instability (MI)

The plane wave

$$u_0(x,t) = Ae^{i\frac{q}{2}A^2 t}$$

is an exact solution of

$$i\partial_t u + \frac{p}{2}\Delta u + \frac{q}{2}|u|^2 u = 0.$$

If we look for a solution of the form

$$u(x,t) = u_0(x,t)(1+\delta(x,t)),$$

we get

$$u_0\left(i\delta_t + \frac{p}{2}\Delta\delta + \frac{q}{2}A^2(\delta+\bar{\delta}) + \frac{q}{2}A^2(\delta+\bar{\delta})\delta + \frac{q}{2}A^2|\delta|^2(1+\delta)\right) = 0.$$

If "$\delta \ll 1$" we can linearise,

$$i\delta_t + \frac{p}{2}\Delta\delta + \frac{q}{2}A^2(\delta+\bar{\delta}) = 0 \quad \Rightarrow \quad \boldsymbol{\delta = e^{i\zeta(x-2\pi\Omega(\zeta)t)}}, \quad \boldsymbol{\Omega^2 = \left(\frac{p}{4\pi}\right)^2\left(\zeta^2 - 2\frac{q}{p}A^2\right)}.$$

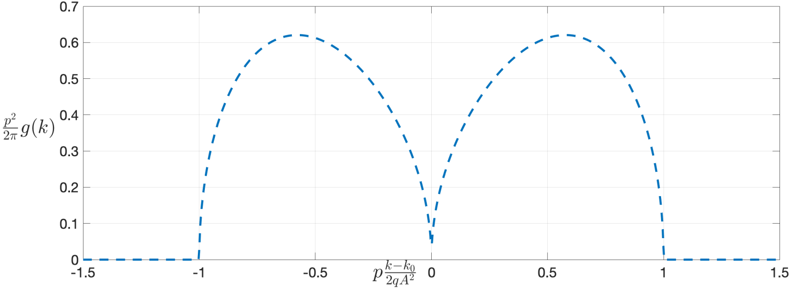
Modulation instability (MI): Linear implications

If u_0 is a plane wave solution of

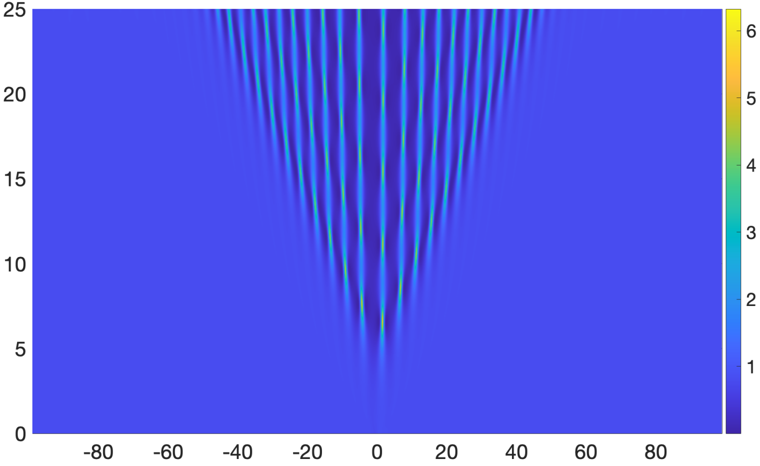
$$i\partial_t u + \frac{p}{2}\Delta u + \frac{q}{2}|u|^2 u = 0,$$

then a perturbation δ will tend to grow exponentially in time, according to

$$\hat{\delta}(k,t) \approx \hat{\delta}(k,0)e^{g(k)t}.$$



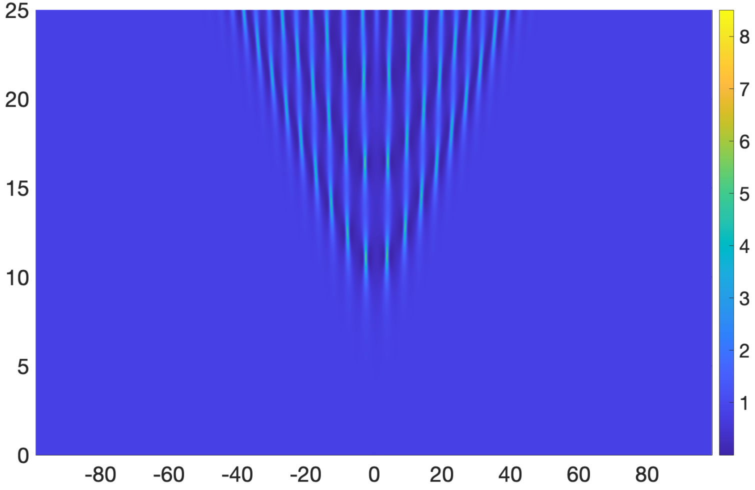
Modulation instability (MI): Nonlinear stage



Computation inspired by

[Biondini & Mantzavinos 2016; Biondini, Mantzavinos & Trillo 2018]

Modulation instability (MI): Nonlinear stage

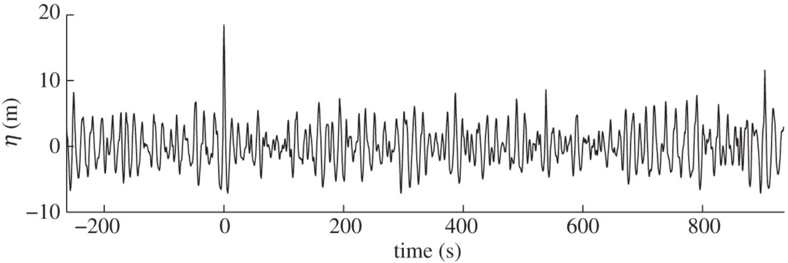


With rough initial inhomogeneity (white noise $\times$ Gaussian envelope), basically same pattern

Rogue Waves: a turning point in the 2000s

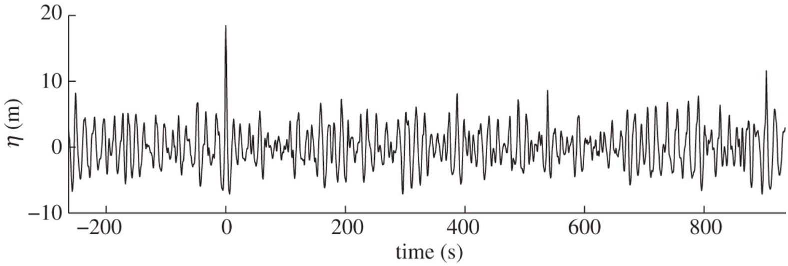
- MV Derbyshire, largest British ship to be lost at sea, in 1980. In 1994-1998 an extensive underwater survey was made. In 2000 the leading explanation that emerged was that it was sank by a *single wave* [Faulkner, SNAME 1999].

- Draupner wave (Jan. 1 1995), 26m trough to peak.
  10,000-40,000 year wave by standard engineering computations
  cf. [Rainey & Colman 2014]



Rogue Waves and the MI

- Does the MI play a significant role in Rogue Waves?
- **Fundamental issue: realistic sea states are not plane waves.**
- **Can we have a "generalised MI" actually applicable to realistic sea states?**



Typical sea states

- Sea states are (locally) stationary and homogeneous random processes characterised by their power spectrum,

$$\mathrm{E}\left[\eta(x,t)\eta(x',t)\right] = \Gamma(x-x') + o(1),$$

$$\mathcal{F}_{y\to k}[\Gamma(y)] = P(k)$$

[Komen et al., *'Dynamics and modelling of ocean waves'*, CUP; Ochi, *'Ocean waves: the stochastic approach'*, CUP]

- This leads to a representation of the form

$$\eta(x,t) = \sum_j Z_j \sqrt{P(k_j)\delta k_j}\,\sin(k_j x - \omega_j t + \phi_j),$$

where Z_j are normal iid RV and ϕ_j are $U(0,2\pi)$ iid RV.

These power spectra form the vast majority of data on ocean waves.

The Alber equation

- Idea: investigate perturbations on

$$\eta(x,t) = \sum_j Z_j \sqrt{P(k_j)\delta k_j} \sin(k_j x - \omega_j t + \phi_j),$$

assuming NLS dynamics and taking ensemble averages.

- I. E. Alber, **Proceedings of the Royal Society A** (1978)
$\sim$ 240 citations in Google scholar

This talk is based on:

- A. Athanassoulis, G. Athanassoulis and T. Sapsis, **Journal of Ocean Engineering and Marine Energy** (2017)
- A. Athanassoulis, **OMAE 2018**
- A. Athanassoulis, G. Athanassoulis, M. Ptashnyk and T. Sapsis, **Kinetic and Related Models** (2020)
- A. Athanassoulis and O. Gramstad, **Fluids** (2021)

Key results

Consider a background solution η_0 with autocorrelation

$$\mathrm{E}[\eta_b(x,t)\eta_b(x',t)] = \Gamma(x-x'),$$

and an inhomogeneity δ such that

$$\mathrm{E}\left[\left(\eta_b(x,0)+\delta_0(x)\right)\left(\eta_b(x',0)+\delta_0(x')\right)\right] = \Gamma(x-x') + \varepsilon\rho(x,x',0).$$

Then $\rho(x,x',t)$ evolves in time under

$$i\partial_t\rho + \frac{p}{2}\left(\Delta_x - \Delta'_x\right)\rho + q\left[\Gamma(x-x') + \varepsilon\rho(x,x')\right]\left[\rho(x,x) - \rho(x',x')\right] = 0$$

- (In)stability condition on $\Gamma(x-x') \Leftrightarrow P(k)$ [Alber, 1978, PRSA]
- Stable case is **Landau damping** [AAPS, 2020, KRM]
- Extrapolating from nonlinear LD results, nonlinear stability is only expected if $\|\delta_0\|_{H^s} \ll 1$.
- In the limit $P(k) \to C\delta(k-k_0)$ we recover the classical MI.

More details **here**.

The Landau-Alber bifurcation

- Stable case: Landau damping

$$\exists \kappa > 0 \qquad \inf_{X, \mathrm{Re}\,\omega > 0} |1 - \widetilde{h}[P](X,\omega)| \geqslant \kappa.$$

and

ρ does not grow under linearised dynamics

- Unstable case: Modulation instability

$$\exists X_* \ \exists\, \mathrm{Re}\,\omega_* \geqslant 0 \qquad \widetilde{h}[P](X_*,\omega_*) = 1$$

and

ρ grows exponentially under linearised dynamics

First constructive way to check stability: [AAPS, 2020, KRM]

Two kinds of functions

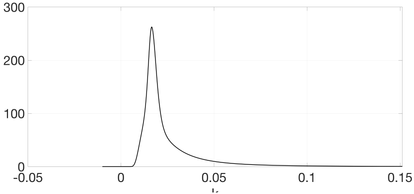
$$\mathrm{E}[\eta(x,t)\eta(x',t)] = \Gamma(x-x') + \varepsilon\rho(x,x',t)$$

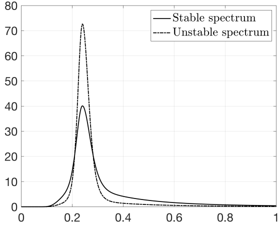
- Solution is broken down to
  spatially homogeneous part + localised part

- Gen. Modulation instability $\Leftrightarrow$ Growth of the localised part
  Landau damping $\Leftrightarrow$ Dispersion of the localised part

- Fine print:
  * timescale & nonlinear evolution
  * smallness of initial localised perturbation for NL LD

The JONSWAP parametric spectrum





The JONSWAP parametric spectrum is defined as

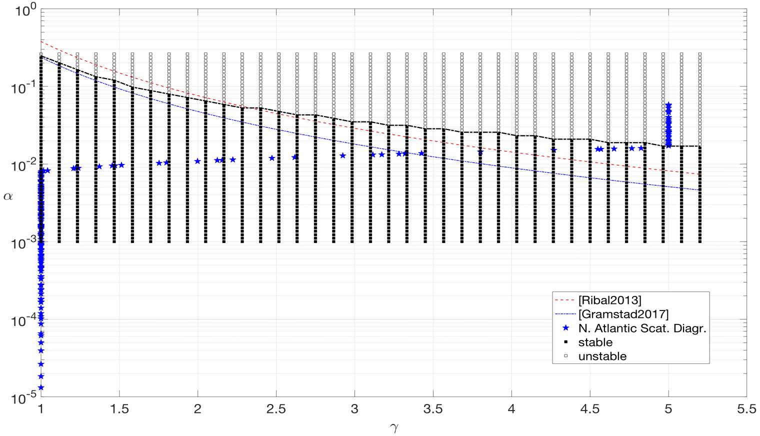
$$P(k) = \frac{\alpha}{2k^3} e^{-\frac{5}{4}\left(\frac{k_0}{k}\right)^2} \gamma^{exp[-(1-\sqrt{k/k_0})^2/2\delta^2]}, \qquad \delta = \delta(k_{rpm}) = \begin{cases} 0.07, & k \leqslant k_0, \\ 0.09, & k > k_0, \end{cases}$$

α : power, H_s

γ : peakedness, narrowbandedness

Stability region for JONSWAP [AAPS, 2020, KRM]

Unstable spectra: $\sim 0.2\%$ of the time.

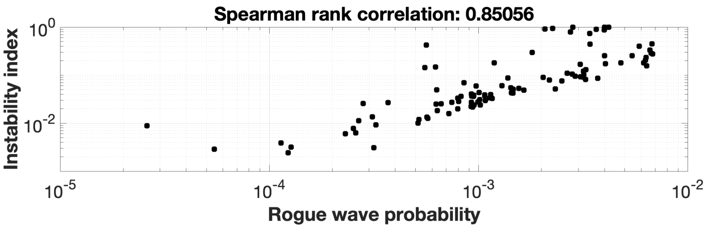


North Atlantic Scatter Diagram data from [DNV-GL, DNVGL-RP-C205: Environmental Conditions and Environmental Loads, Tech. Rep., August 2017]

Quantifying the onset of MI in realistic sea states

- **Phase-resolved, fully nonlinear simulations** can put the predictions of Alber equation to the test.

- **Size of events due to MI now quantifiable**
  $\Rightarrow$ **Not all extreme events are due to MI**



[Janssen, (2003); A. & Gramstad, Fluids (2021)]

What is MI? What is LD? Follow the theory.

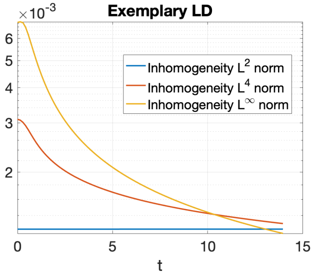
Consider the "background solution" of

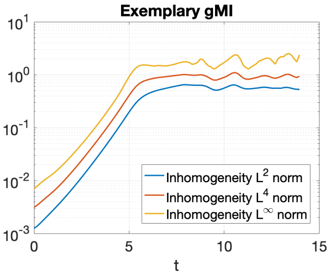
$$iu_t + \frac{p}{2}\Delta u + \frac{q}{2}|u|^2 u = 0, \qquad u(x,0) = u_0(x) = \sum A_j e^{2\pi i [k_j x + \phi_j]}$$

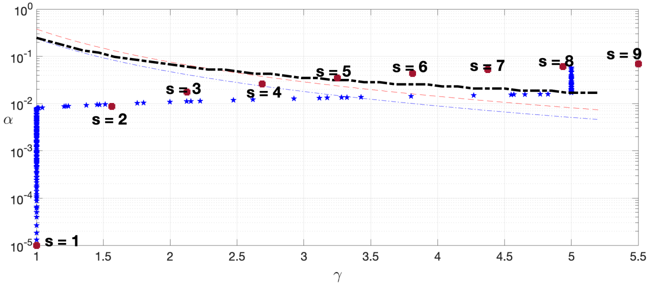
and the perturbed solution

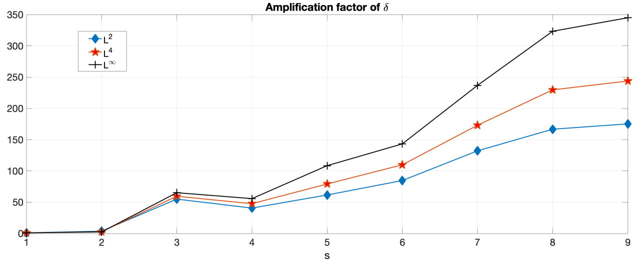
$$iv_t + \frac{p}{2}\Delta v + \frac{q}{2}|v|^2 v = 0, \qquad v(x,0) = u_0(x) + \delta_0(x)$$

where δ_0 is a small inhomogeneity.



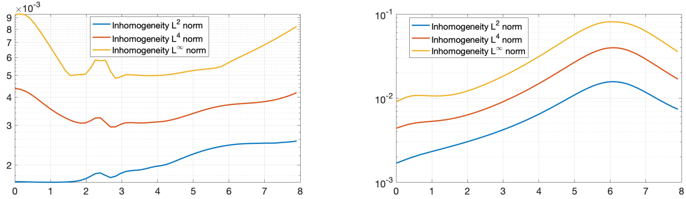



s = 1
s = 2
s = 3
s = 4
s = 5
s = 6
s = 7
s = 8
s = 9
α
γ


Amplification factor of δ
L^2
L^4
L^∞
s

Beyond LD

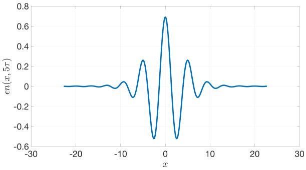
- In the "stable" case we expect LD for the linearised problem
- Nonlinear LD typically requires $\frac{\|\delta_0\|_{H^s}}{\kappa^2}$ small enough



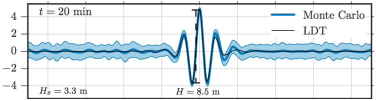
- κ scales roughly with how far the spectrum is from being unstable
- Hence LD does not apply to spectra close to instability
  Still, inhomogeneity plateaus to something small

Conclusions

- Deep in the unstable regime, MI *inevitably* yield large events
- Not all large events are due to MI – especially for stable spectra
- **More extensive Monte Carlo & crossing seas extensions warranted**



[A., Athanassoulis, Sapsis, JOEME (2017)]
Scalings of unstable modes



[Dematteis, Grafke, Vanden-Eijden, PNAS (2018)]
Monte Carlo + Large Deviations Theory
for fully Nonlinear simulations of mNLS

Thank you for your attention!

Details of the computation

- Initial data $u_0(x)$:

$$u_0(x) = \sum_j Z_j \cdot \sqrt{P(k_j)\delta k_j} \;\cdot\; e^{k_j x - \omega_j t}$$

- $\sqrt{P(k_j)\delta k_j}$: average power associated with $k \in (k_j, k_{j+1})$
- Z_j : randomised coefficients $\frac{1}{\sqrt{2}}(\mathcal{N}(0,1) + i\mathcal{N}(0,1))$
- Industry common practice: average over $\sim$ 500 realisations ...
- Here: Keep same realisation of Z_j, vary α, γ.
- Here: a handful of realisations so far

Working with the (in)stability condition

How do you check

$$\exists \kappa > 0 \qquad \inf_{X, \operatorname{Re}\omega > 0} |1 - \widetilde{h}(X, \omega)| \geqslant \kappa?$$

More or less through

$$\exists X_* \ \exists \operatorname{Re}\omega_* \geqslant 0 \qquad \widetilde{h}(X_*, \omega_*) = 1$$

A **nonlinear system of two equations in three unknowns**,

$$\operatorname{Re}\left(\widetilde{h}(X_*, a_* + ib_*)\right) = 1 \quad \text{and} \quad \operatorname{Im}\left(\widetilde{h}(X_*, a_* + ib_*)\right) = 0,$$

where of course $\omega_* = a_* + ib_*$.

Denote by $\mathbb{H}[f](t)$ the Hilbert transform,

$$\mathbb{H}[f](t) := p.v. \frac{1}{\pi} \int \frac{f(x)}{t-x} dx.$$

Theorem [A., Athanassoulis, Ptashnyk & Sapsis, KRM (2020)]

The following are equivalent:

(1). $\inf_{\substack{\mathrm{Re}\,\omega>0,\\ X\in\mathbb{R}}} |1-\widetilde{h}(X,\omega)| = 0$

(2). $\exists\ X_* \in \mathbb{R}, \quad \Omega_* \in \mathbb{C}\backslash\mathbb{R} \quad : \quad \mathbb{H}[D_{X_*}P](\Omega_*) = \mathbb{H}[D_{X_*}P](\overline{\Omega}_*) = \frac{4\pi p}{q}$

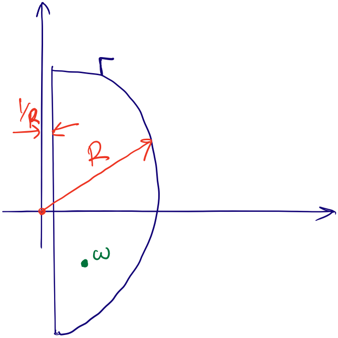
or

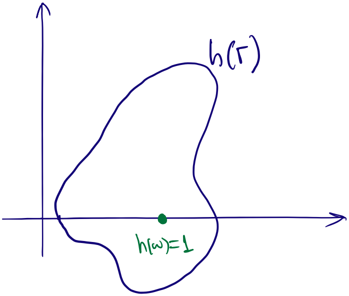
$\exists\ X_*, \Omega_* \in \mathbb{R} \quad : \quad \mathbb{H}[D_{X_*}P](\Omega_*) = \frac{4\pi p}{q}$ and $D_{X_*}P(\Omega_*) = 0.$

(3). $d(\overline{\Gamma}, 4\pi p/q) = 0$, where $\mathbb{S}[f](t) = \mathbb{H}[f](t) - if(t)$ and

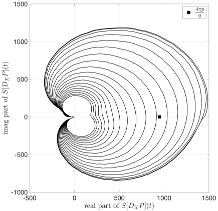
$$\Gamma_X := \{\mathbb{S}[D_X P(\cdot)](t),\ t \in \mathbb{R}\} \cup \{0\}, \qquad \overset{\circ}{\Gamma}_X = \{z \in \mathbb{C} | z \text{ enclosed by } \Gamma_X\},$$

$$\overline{\Gamma} := \overline{\bigcup_{X\in\mathbb{R}} \overset{\circ}{\Gamma}_X}.$$

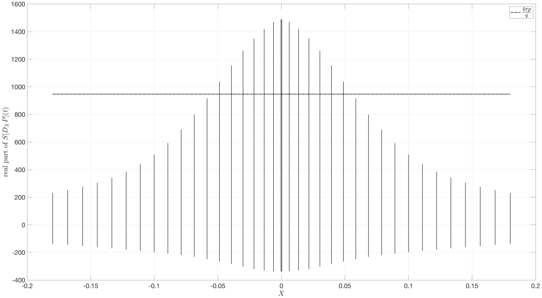




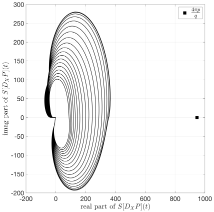
Γ winds around ω $\Rightarrow$ $h(\Gamma)$ winds around $h(\omega)$



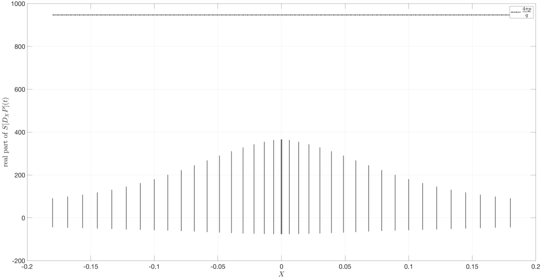
$\frac{4\pi p}{q}$
imag part of $S[D_X P](t)$
real part of $S[D_X P](t)$



$\frac{4\pi p}{q}$
real part of $S[D_X P](t)$
X



$\frac{4\pi p}{q}$
imag part of $S[D_X P](t)$
real part of $S[D_X P](t)$

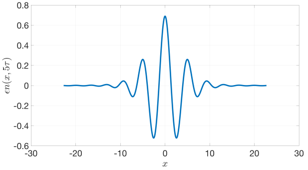


$\frac{4\pi p}{q}$
real part of $S[D_X P](t)$
X

If there are unstable wavenumbers,

$$\widetilde{h}(X_*, \omega_*) = 1 \qquad \text{for some } X_0 \in \mathbb{R},\ \operatorname{Re}\omega_* > 0$$

they give rise to **unstable modes**. These seem to successfully capture some inherent scalings for Rogue waves.



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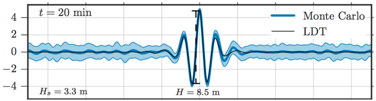


[Dematteis, Grafke, Vanden-Eijden, PNAS (2018)]
Monte Carlo + Large Deviations Theory
for fully Nonlinear simulations of mNLS

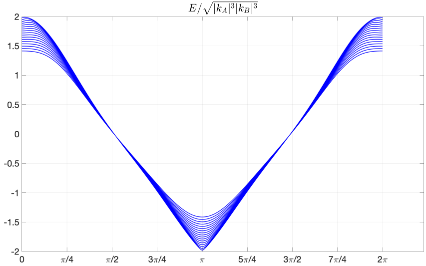
$\Rightarrow$ Timescales $\Rightarrow$ Unstable spectra $\begin{cases} \text{in real life?} \\ \text{in the Alber equation?} \end{cases}$

Crossing seas

NLS system for crossing seas [Gramstad & Trulsen, Physics of Fluids, 2011]

$$i\frac{d}{dt}\tilde{v}^A - \frac{\omega_A}{8k_A^4}\left(k_{Ax}^2\partial_x^2 - 2k_{Ax}^2\partial_y^2\right)\tilde{v}^A - \left(k_A^3|\tilde{v}^A|^2 + E|\tilde{v}^B|^2\right)\tilde{v}^A = 0,$$

$$i\frac{d}{dt}\tilde{v}^B - \frac{\omega_A}{8k_B^4}\left((k_{Bx}^2 - 2k_{By}^2)\partial_x^2 + (k_{By}^2 - 2k_{Bx}^2)\partial_y^2 + 6k_{Bx}k_{By}\partial_x\partial_y\right)\tilde{v}^B -$$
$$- \left(k_B^3|\tilde{v}^B|^2 + E|\tilde{v}^A|^2\right)\tilde{v}^B = 0,$$



Crossing seas

$$i\frac{d}{dt}\tilde{v}^A - \frac{\omega_A}{8k_A^4}\left(k_{Ax}^2\partial_x^2 - 2k_{Ax}^2\partial_y^2\right)\tilde{v}^A - \left(k_A^3|\tilde{v}^A|^2 + E|\tilde{v}^B|^2\right)\tilde{v}^A = 0,$$

$$i\frac{d}{dt}\tilde{v}^B - \frac{\omega_A}{8k_B^4}\left((k_{Bx}^2 - 2k_{By}^2)\partial_x^2 + (k_{By}^2 - 2k_{Bx}^2)\partial_y^2 + 6k_{Bx}k_{By}\partial_x\partial_y\right)\tilde{v}^B - \\ - \left(k_B^3|\tilde{v}^B|^2 + E|\tilde{v}^A|^2\right)\tilde{v}^B = 0,$$

Leads to an Alber system with the following

Penrose condition:

$\exists \mathbf{P} \in \mathbb{R}^2,\ \omega \in \mathbb{C}$ such that

$$\inf_{\substack{\mathrm{Re}\,\omega>0 \\ \mathbf{X}\in\mathbb{R}^2}} \left|(1 - k_A^3 h^A(\mathbf{X},\omega))(1 - k_B^3 h^B(\mathbf{X},\omega)) - E^2 h^A(\mathbf{X},\omega)h^B(\mathbf{X},\omega)\right| = \kappa > 0$$

where

$$h^A(\mathbf{X},\omega) \sim \mathbb{H}[\int_\perp D_X P^A](\frac{\omega}{M(\mathbf{X})}), \qquad \text{similarly for } h^B(\mathbf{X},\omega)$$