Uniqueness of Regular Exact Borel Subalgebras

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Introduction

- $\mathfrak{g}$ f.d. complex semisimple Lie algebra.
- Category $\mathcal{O}$ is the category of certain representations of $\mathfrak{g}$.
- Category $\mathcal{O}$ is a highest weight category, the standard objects are the duals of the Verma modules.
- Blocks of category $\mathcal{O}$ are module categories of quasi-hereditary algebras, the standard modules are the Verma modules.

Quasi-Hereditary Algebras

Definition

[Cline, Parshall, Scott]

- A finite-dimensional algebra.
- $\mathrm{Sim}(A)$ isomorphism classes of simple A-modules.
- $\leq$ partial order on $\mathrm{Sim}(A)$.
- Projective covers $P(L)$ for $L \in \mathrm{Sim}(A)$.

Then the **standard modules** are defined as

$$\Delta(L) := P(L) / \sum_{L' \not\leq L, \varphi : P(L') \to P(L)} \mathrm{im}(\varphi), \; L \in \mathrm{Sim}(A).$$

$F(\Delta)$ is the category of standardly filtered modules.
A is called **quasi-hereditary** if $\mathrm{End}_A(\Delta(L)) \cong \mathrm{k}$ for all $L \in \mathrm{Sim}(A)$ and $A \in \mathrm{F}(\Delta)$.

Example: $\mathcal{O}_0(\mathfrak{sl}_2)$

Let Q be the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

and let $A := \mathrm{k}\, Q/(\alpha\beta)$.

$$P(L_1) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}; \Delta(L_1) = \begin{pmatrix} 1 \end{pmatrix} = L_1; P(L_2) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \Delta(L_2).$$

So $P(L_1)$ has a Δ-filtration $P(L_1) = \begin{pmatrix} \Delta(L_1) \\ \Delta(L_2) \end{pmatrix}$ and $P(L_2) = \Delta(L_2)$, and $A \cong P(L_1) \oplus P(L_2)$.

Borel Subalgebras

- $\mathfrak{g}$ has a Borel subalgebra $\mathfrak{b}$.
- Verma modules are simple modules induced along the Borel subalgebra:

$$V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

Exact Borel Subalgebras

Definition

[Koenig] An **exact Borel subalgebra** B is a subalgebra of a quasi-hereditary algebra A such that

- $A \otimes_B -$ is exact.
- There is a bijection $\varphi : \mathrm{Sim}(B) \to \mathrm{Sim}(A)$ such that $A \otimes_B L \cong \Delta(\varphi(L))$ for all $L \in \mathrm{Sim}(B)$.
- B is directed, i.e. $\mathrm{Ext}^1_B(L, L') \neq (0)$ implies $\varphi(L) < \varphi(L')$.

B is called regular if for all $L, L' \in \mathrm{Sim}(B)$ the map

$$\mathrm{Ext}^n_B(L, L') \to \mathrm{Ext}^n_A(\Delta(L), \Delta(L')), [f] \mapsto [A \otimes_B f]$$

is an isomorphism for $n \geq 1$.

Example

In the example where Q is

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd}$$

and $A := \mathrm{k}\, Q/(\alpha\beta)$, the subalgebra given by the quiver

$$1 \xrightarrow{\alpha} 2$$

is a regular exact Borel subalgebra.

Existence of Regular Exact Borel Subalgebras

In general, a quasi-hereditary algebra does not have an exact Borel subalgebra!

Theorem

[Koenig, Külshammer, Ovsienko] Let $(A, \leq_A)$ *be a quasi-hereditary algebra. Then there is a Morita equivalent quasi-hereditary algebra* $(R, \leq_R)$ *which has a basic regular exact Borel subalgebra* B*.*

Uniqueness of Regular Exact Borel Subalgebras

The Borel subalgebra $\mathfrak{b}$ of a Lie algebra $\mathfrak{g}$ is unique up to inner automorphism.

Theorem

[Külshammer, Miemietz] Let A be a quasi-hereditary algebra and suppose B and B' are basic regular exact Borel subalgebras of A. Then there is an automorphism $\varphi : A \to A$ such that $\varphi(B') = B$.

Related results obtained in [Conde].

Q: Can we choose φ to be an inner automorphism?

A: Yes! We can even do this in a slightly more general setting.

Setup

- k algebraically closed field.
- A finite-dimensional algebra over k.
- $M_1, \ldots, M_n$ pairwise non-isomorphic indecomposable modules in mod A, $M := \bigoplus_{i=1}^n M_i$.
- $\mathrm{F}(M)$ subcategory of mod A consisting of modules admitting a filtration by $M_1, \ldots, M_n$.

Bound Quivers

Definition

[Külshammer] A **bound quiver** for $M_1, \ldots, M_n$ is a basic subalgebra B of A s.t.

- $A \otimes_B - : \operatorname{mod} B \to \operatorname{mod} A$ is exact.
- $\operatorname{Sim}(B) = \{L_1^B, \ldots, L_n^B\}$
- $A \otimes_B L_i^B \cong M_i$.
- For $k \geq 1$, $1 \leq i, j \leq n$

$$\operatorname{Ext}_B^k(L_i^B, L_j^B) \to \operatorname{Ext}_A^k(M_i, M_j), [f] \mapsto [\mathrm{id}_A \otimes f]$$

is an isomorphism.

A-infinity Algebras

Definition

An **A-infinity algebra** over k is a graded vector space $\mathcal{A}$ together with maps

$$m_n : \mathcal{A}^{\otimes_{\mathsf{k}} n} \to \mathcal{A}$$

for all $n \geq 1$, homogeneous of degree $2 - n$, such that for all $n \in \mathbb{N}$

$$\sum_{r+s+t=n, s\geq 1} (-1)^{rs+t} m_n(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.$$

An **A-infinity category** is a generalization of this to multiple objects.

Examples of A-infinity Algebras

- A f.d. k-algebra is an A-infinity algebra with $m_n = 0$ for $n \neq 2$. Similarly, a category is an A-infinity category with $m_n = 0$ for $n \neq 2$.
- A dg-algebra $\mathcal{A}$ over k is an A-infinity algebra with $m_n = 0$ for $n \neq 1, 2$. Similarly, a dg-category is an A-infinity category with $m_n = 0$ for $n \neq 2$.
- If $\mathcal{A}$ is an A-infinity algebra with $m_1 = 0$, then $(\mathcal{A}, m_2)$ is a graded algebra and $(\mathcal{A}_0, m_2)$ is an algebra. Similarly, if $\mathcal{A}$ is an A-infinity category with $m_1 = 0$, then $(\mathcal{A}, m_2)$ is a graded category and $(\mathcal{A}_0, m_2)$ is a usual category.

A-infinity homomorphism

Definition

An **A-infinity homomorphism** $f : \mathcal{A} \to \mathcal{B}$ is a family $f = (f_n)_n$ of k-linear maps

$$f_n : \mathcal{A}^{\otimes n} \to \mathcal{B}$$

homogeneous of degree $1 - n$ such that for all $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \sum_{j_1+\cdots+j_k=n} (-1)^{?} m_k(f_{j_1} \otimes \cdots \otimes f_{j_n})$$
$$= \sum_{r+s+t=n, s\geq 1} (-1)^{?} f_{r+t+1}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}).$$

f is called **strict** if $f_n = 0$ for $n > 1$.

Unitality

Definition

An A-infinity algebra $\mathcal{A}$ over k is called strictly unital over a k-algebra L if there is a strict A-infinity homomorphism $f = (f_n)_n : L \to \mathcal{A}$ such that

$$m_n(a_1 \otimes \cdots \otimes a_n) = 0$$

for all $n \neq 2$, $a_1, \dots a_n \in \mathcal{A}$ s.t. $a_i \in f_1(L)$ for some $1 \leq i \leq n$, and

$$m_2(f_1(1_L) \otimes a) = a = m_2(f_1(1_L) \otimes a)$$

for all $a \in \mathcal{A}$.

In our case: $L = \mathrm{k}^n$.

A-infinity structures on Ext-algebras

Theorem

(Kadeishvili) If $\mathcal{A}$ is an A-infinity algebra strictly unital over some semisimple subalgebra L, then $H^*(\mathcal{A})$ has the structure of an A-infinity algebra strictly unital over L, with $m_1^{H^*(\mathcal{A})} = 0$ and $m_2^{H^*(\mathcal{A})}$ being induced by $m_2^{\mathcal{A}}$; and there is a strictly unital A-infinity quasi-isomorphism $(f_n)_n : H^*(\mathcal{A}) \to \mathcal{A}$ such that $H^*(f_1) = \mathrm{id}_{H^*(\mathcal{A})}$.

Remark

This also works for A-infinity categories.

Corollary

Let $M_1, \ldots, M_n$ be a collection of objects in $\operatorname{mod} A$, and $M := \bigoplus_{i=1}^n M_n$. Then $\operatorname{Ext}_A^*(M, M)$ has the structure of an A-infinity algebra strictly unital over k^n.

Twisted Module Category

Definition

[Bondal, Kapranov] Let $\mathcal{A}$ be an A-infinity algebra strictly unital over L.
The **twisted module category** $\mathrm{twmod}_L(\mathcal{A})$ is given by:

- Objects: Pairs (X, w_X); X L-module, $w_X \in \mathcal{A}_1 \otimes_{L \otimes L^{\mathrm{op}}} \mathrm{End}_{\mathrm{k}}(X)$ s.t.

$$\sum_{n=1}^{\infty} (-1)^? m_n(w_X^{\otimes n}) = 0$$

- Morphism spaces: $\mathcal{A} \otimes_{L \otimes L^{\mathrm{op}}} \mathrm{Hom}_{\mathrm{k}}(X, Y)$.
- Higher multiplications in $\mathrm{twmod}_L(\mathcal{A})$: Higher multiplications of $\mathcal{A}$ twisted by w_X.

Twisted Module Categories and Filtered Modules

Remark

In particular $H^*(\mathrm{twmod}_L(\mathcal{A}))$ *obtains the structure of an A-infinity category with* $m_1 = 0$. *Thus,* $H^0(\mathrm{twmod}_L(\mathcal{A}))$ *viewed with composition given by* m_2 *has the structure of a usual category.*

Lemma

[Seidel] This extends to a functor $H^0(\mathrm{twmod}_L)$ *from the category of A-infinity algebras an homomorphisms strictly unital over* L *to* Cat

Theorem

[Keller] There is an equivalence

$$R_M : H^0(\mathrm{twmod}_L(\mathrm{Ext}^*_A(M, M))) \cong \mathrm{F}(M).$$

Back to Bound Quivers

Let B be a bound quiver for (A, M). Then we obtain an A-infinity homomorphism

$$f^B : \mathrm{Ext}^*_B(L^B, L^B) \to \mathrm{Ext}^*_A(M, M)$$

such that f_1^B is an isomorphisms in degree > 0.
Moreover, we have equivalences

$$R_{L^B} : H^0(\mathrm{twmod}_L(\mathrm{Ext}^*_B(L^B, L^B))) \to \mathsf{F}(L^B) = \mathrm{mod}\, B$$

and

$$R_M : H^0(\mathrm{twmod}_L(\mathrm{Ext}^*_A(M, M))) \to \mathsf{F}(M) \subseteq \mathrm{mod}\, A.$$

A Commutative Diagram

Proposition

[RR] The diagram

$$\begin{tikzcd}
H^0(\mathrm{twmod}_L(\mathrm{Ext}^*_B(L^B, L^B))) \arrow[r, "R_{L^B}"] \arrow[d, "H^0(\mathrm{twmod}_L(f^B))"'] & \mathrm{mod}\, B \arrow[d, "A \otimes_B -"] \\
H^0(\mathrm{twmod}_L(\mathrm{Ext}^*_A(M, M))) \arrow[r, "R_M"] & \mathrm{F}(M)
\end{tikzcd}$$

commutes up to natural isomorphism.

Proposition

[RR] Let B and B' be bound quivers for (A, M). Then, there is A-infinity isomorphism

$$h : \mathrm{Ext}^*_{B'}(L^{B'}, L^{B'}) \to \mathrm{Ext}^*_B(L^B, L^B)$$

such that the diagram

$$\begin{tikzcd}
\mathrm{Ext}^*_B(L^B, L^B) \arrow[dr, "f^B"] & \\
& \mathrm{Ext}^*_A(M, M) \\
\mathrm{Ext}^*_{B'}(L^{B'}, L^{B'}) \arrow[uu, "h"] \arrow[ur, "f^{B'}"] &
\end{tikzcd}$$

commutes.

Corollary

There is an equivalence $H : \operatorname{mod} B' \to \operatorname{mod} B$ *such that the diagram*

$$\begin{tikzcd}
\operatorname{mod} B \arrow[dr, "A \otimes_B -"] & \\
& \mathsf{F}(M) \\
\operatorname{mod} B' \arrow[uu, "H"] \arrow[ur, "A \otimes_{B'} -"'] &
\end{tikzcd}$$

commutes up to natural isomorphism.

The Main Theorem

Theorem

[RR] Let B and B' be bound quivers for (A, M). Then there is $a \in A^\times$ such that $B' = aBa^{-1}$.

Proof Sketch:
B' basic projective generator in mod B'
$\Rightarrow H(B')$ basic projective generator in mod B
$\Leftrightarrow H(B') \cong B$.
Changing H on one object, we can assume $H(B') = B$.
Denote by $\alpha : H' \to (A \otimes_B -) \circ H$ the natural isomorphism.
Then $\alpha_{B'} : A \otimes_{B'} B' \to A \otimes_B B$ isomorphism.

Obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_{B'}(B') \arrow[r, "\mathrm{id}_A \otimes -"] \arrow[d, "f \mapsto H(f)"'] & \mathrm{End}_A(A \otimes_{B'} B') \arrow[r, "\rho_{e'}"] \arrow[d, "\rho_{\alpha_{B'}}"'] & \mathrm{End}_A(A) \arrow[d, dashed, "\rho_\beta"'] \\
\mathrm{End}_B(B) \arrow[r, "\mathrm{id}_A \otimes -"] & \mathrm{End}_A(A \otimes_B B) \arrow[r, "\rho_e"] & \mathrm{End}_A(A)
\end{tikzcd}$$

where

$$e : A \otimes_B B \to A, a \otimes b \mapsto ab$$
$$e' : A \otimes_{B'} B' \to A, a \otimes b' \mapsto ab'$$

and ρ denotes conjugation. $\beta \in \mathrm{End}_A(A)^\times \Rightarrow \beta = r_a$, $a \in A^\times$.

$$\begin{tikzcd}
(B')^{\mathrm{op}} \arrow[d] \arrow[drrr, bend left=10, "\iota_{B'}"'] & & & \\
\mathrm{End}_{B'}(B') \arrow[r] & \mathrm{End}_A(A \otimes_{B'} B') \arrow[r] & \mathrm{End}_A(A) \arrow[r] \arrow[d, "\rho_{r_a}"'] & A \arrow[d, "\rho_{a-1}"'] \\
\mathrm{End}_B(B) \arrow[r] & \mathrm{End}_A(A \otimes_B B) \arrow[r] & \mathrm{End}_A(A) \arrow[r] & A \\
(B)^{\mathrm{op}} \arrow[u] \arrow[urrr, bend right=10, "\iota_B"] & & &
\end{tikzcd}$$

Thank you!

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