Introduction to Symplectic Cohomology and Applications

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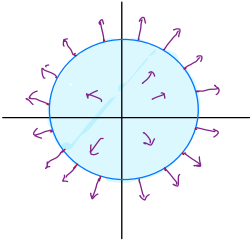
Underlying Symplectic Structure

We consider certain noncompact symplectic manifolds.

A Liouville domain

$=$ A compact symplectic manifold $\underline{M}$ with boundary $\partial\underline{M}$ with

- exact symplectic 2-from: $\omega = d\alpha$.
- Liouville vector field X:
  - $\iota_X d\alpha_M = \alpha_M \iff \mathcal{L}_X d\alpha = d\alpha$.
  - X is transversely pointing outwards along ∂M.



$(r, \theta) \in \mathbb{C}$

$\alpha = \frac{1}{2} r^2 d\theta$

$d\alpha = r \cdot dr\, d\theta$

$\iota_Z d\alpha = \alpha$

$Z = \frac{1}{2} r \cdot \frac{\partial}{\partial r}$

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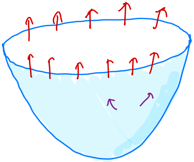
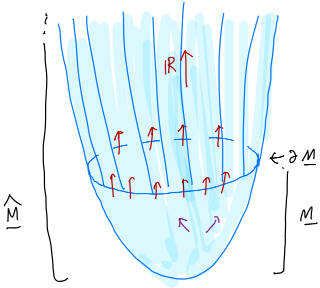


Completion of a Liouville Domain

Completion of Liouville domain $\widehat{\underline{M}}$
$=$ Half-infinite cylinder attached to ∂M to complete the Liouville flow.

$$\widehat{\underline{M}} := \left(\underline{M} \cup (\partial \underline{M} \times [0, \infty)), d(e^r \alpha)\right).$$



Example of Liouville Domain

Affine Variety M
$=$ A set of solution of system of polynomial equations in $\mathbb{C}$.

By intersecting large radious ball centered at 0 with M, we get a Liouville domain $\underline{M}$. Moreover, the completion of $\underline{M}$ is symplectomorphic to M.

- $M \hookrightarrow \mathbb{C}^N$ algebraic embedding as variety.
  Consider polar coordinates (r_i, θ_i) in $\mathbb{C}^N$. Define $f := \frac{1}{4}\Sigma r_i^2$.
  Then $dd^c f = \Sigma r dr \wedge d\theta$ exhausting plurisubharmonic of finite type. $\Rightarrow \underline{M}$.
- Consider a line bundle L with a section s satisfying $s^{-1}(0) = D$. Define $g := -log||s||$ and $\omega := -dd^c g \Rightarrow \overline{M}$.
- $\underline{M}$, $\overline{M}$ are Liouville deformation equivalent.

Example of (Completion of) Liouville Domain

Affine Variety M
$=$ A set of solution of system of polynomial equations in $\mathbb{C}$.

$\subset$ An example of Stein manifold of finite type
$=$ A complex manifold (M, J), properly embedded in $\mathbb{C}^N$ with a function $f : M \to \mathbb{R}$ satisfying,

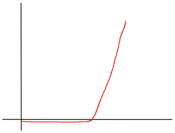
- (Plurisubharmonic) $(-dd^c f)(v, Jv) > 0$ for all $v \neq 0$,
- (Exhausting) $f : M \to \mathbb{R}$ is bounded below and the preimage of every compact set is compact,
- (Of finite type) f has only finitely many singularities.

Given an affine variety M, we get an associated **Liouville domain** $\underline{M}$ **by intersecting large radious ball with** M. Take a **"gradient-like" vector field of** f as our Liouville vector field.

Affine Variety $\subset$ **Completion of a Liouville domain**.

Hamiltonian Vector Field

- $H : \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$, a time-dependent Hamiltonian linear at ∞



- Hamiltonian vector field X_H : $\iota_{X_H}\omega = -dH$
- Hamiltonian 1-periodic orbit
  $x : S^1 = \mathbb{R}/\mathbb{Z} \to (M, \omega)$ with $\dot{x}(t) = X_{H_t}(x(t))$.

Hamiltonian Vector Field

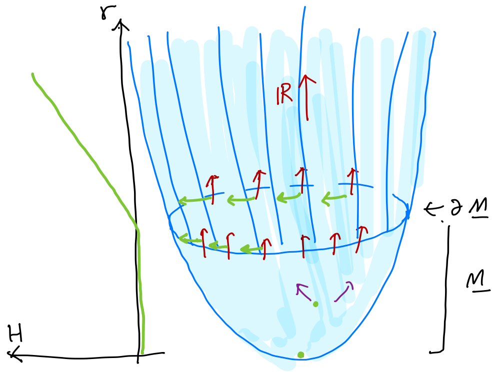


Figure: Liouville Domain Example

Hamiltonian Vector Field

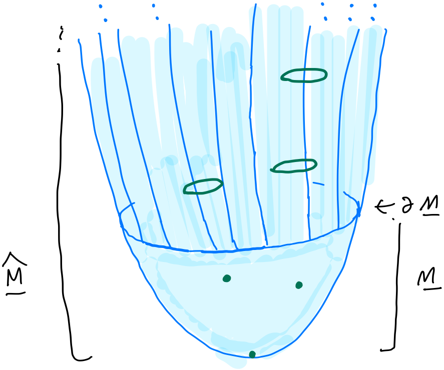


Figure: Liouville Domain Example

Arnold's Conjecture and Floer's Answer

Conjecture (Arnold)

The minimun number of fixed points for a Hamiltonian symplectomorphism $\phi : (M, \omega) \to (M, \omega)$ is bounded from below by the the sum of Betti numbers of M.

$$\#Fix(\phi) \geq rank(H^*(M)).$$

Theorem (Floer)

The conjecture holds under certain assumptions.

on Conley, Zehnder, Gromov, Donaldson, Taubes, Uhlenbeck, Witten...

Hamiltonian Floer Cohomology

Hamiltonian Floer Cohomology $\sim$ Morse Theory on Loop space.

- Symplectic Action Functional $\sim$ "Height" on $\mathcal{L}M$:
  for a loop $x : S^1 \to (M, \omega)$

  $$\mathcal{A}_H(x) := -\int_{S^1} x^*\theta + \int_0^1 H(t, x(t))dt$$

- $d\mathcal{A}_H(x) \cdot (\xi) = -\int_0^1 \omega(\xi, \dot{x} - X_H)dt = 0,$

  $x \in \mathcal{L}_0 M, \xi \in T_x\mathcal{L}M.$

- Gradient flow line, $u : \mathbb{R} \times S^1 \to M$ satisfying $(s, t) \in \mathbb{R} \times S^1$

  $$\partial_s u = -\nabla \mathcal{A}_{H_t}(u) \iff \partial_s u + J_t(\partial_t u - X_{H_t}) = 0.$$

  $$\because \int_0^1 g_t(\eta, (\nabla \mathcal{A}_H)_x) = (d\mathcal{A})_x(\eta) = -\int_0^1 \omega(\eta, \dot{x} - X_H)$$
  $$= \int_0^1 g_t(\eta, J_t(\dot{x} - X_H))$$

Hamiltonian Floer Cohomology

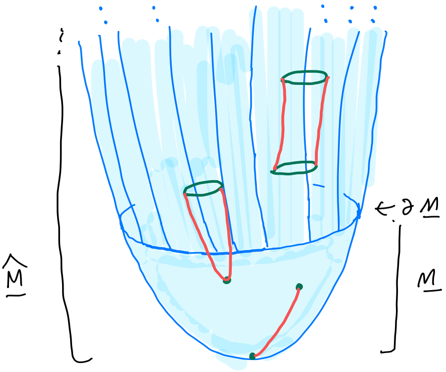


Figure: Liouville Domain Example

Example of Maximum Principle

A hypersurface $Y \subset M$ of a Liouville manifold is called J-convex if locally regular level set of a pluri-subharmonic function $dd^c\phi \leq 0$, where $d^c := J^* d$.

Lemma

Let $Y \subset M$ be a J-convex hypersurface of a Liouville manifold with a plurisubharmonic function f. Then no J-holomorphic curve $u : \mathbb{R} \times S^1 \to M$ can have an interior tangency point with Y (i.e., $f \circ u$ can not have a maximum interior of the domain).

$$
\begin{aligned}
& -\Delta(f \circ u) ds \wedge dt \\
&= dd^c_{J_0}(f \circ u) \\
&= dJ_0^* u^* df = du^* J^* df \\
&= u^* dd^c_J f \leq 0
\end{aligned}
$$

Symplectic Cohomology

$HF^*(M, H_\lambda)$

- Generators: Hamiltonian 1-orbit, $x : S^1 \to (M, \omega)$.
- Differential: Floer cylinders connecting two critical points $x_\pm$, $u : \mathbb{R} \times S^1 \to M$

Symplectic Cohomology of Completion of a Liouville Domain

$$SH^*(M) := \lim_{\lambda \to \infty} HF^*(M, H_\lambda)$$

The action functional $\mathcal{A}_{H_t}(u(s,t))$ increases in s along the gradient flow, since $\partial_s(\mathcal{A}_{H_t}(u(s,t))) = d\mathcal{A}_{H_t} \cdot \partial_s u$ $= -\int_0^1 \omega(\partial_s u, \partial_t u - X_{H_t})dt = -\int_0^1 |\partial_s u|^2_{g_t} dt < 0$.

The action filtration on symplectic cochains induces long exact sequences, for small $\epsilon > 0$.

Long Exact Sequence by the Action Filtration

The action filtration on symplectic cochains induces long exact sequences, for small $\epsilon > 0$,

$$
\begin{tikzcd}
\to SH^*_{[-\epsilon,\epsilon)}(M) \arrow[r] & SH^*_{[-\epsilon,\infty)}(M) \arrow[r] & SH^*_{[\epsilon,\infty)}(M) \arrow[dll, rounded corners, "\delta"] \\
SH^{*+1}_{[-\epsilon,\epsilon)}(M) \arrow[r] \arrow[d, "\cong"] & SH^{*+1}_{[-\epsilon,\infty)}(M) \arrow[r] & SH^{*+1}_{[\epsilon,\infty)}(M) \to \\
H^{*+1}(M) & &
\end{tikzcd}
$$

Long Exact Sequence by the Action Filtration

The action filtration on symplectic cochains induces long exact sequences, for small $\epsilon > 0$,

$$\cdots \to H^*(M) \longrightarrow SH^*(M) \longrightarrow SH^*_+(M) \xrightarrow{\delta} H^{*+1}(M) \longrightarrow SH^{*+1}(M) \longrightarrow SH^{*+1}_+(M) \to \cdots$$

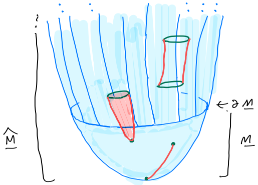


Figure: Floer Cylinders

Motivation

Definition

- A projective variety X over $\mathbb{C}$ is uniruled if for a generic point $x \in X$, $\exists$ a rational map $\mathbb{C}P^1 \to X$ passing through x.
- An affine variety M over $\mathbb{C}$ is $\mathbb{C}$-uniruled if for a generic point $x \in M$, $\exists$ a polynomial map $\mathbb{C} \to M$ passing through x.
- E.g. Exceptional locus of a blow-up.

Mori's Minimal Model Program $\sim$ Study of Rational Curves on a Variety

- Minimal Model: $\mathcal{K}_X = \wedge^{top} T_X^*$ is nef ($\mathcal{K}_X.C \geq 0$).
- Rational curves C with $\mathcal{K}_X.C < 0$ is an obstruction for $\mathcal{K}_X$ to be nef.
- Goal of MMP: Get rid of some rational curves. Classify.

Main Theorem

Symplectic Criteria on Stratified Uniruledness of Affine Variety

$$\cdots \to SH^{m-1}(M) \to SH_+^{m-1}(M) \longrightarrow H^m(M) \to SH^m(M) \to \cdots$$

If there exist $[\mho] \in H^m(M)$ that is the image of δ for $m = 2k$ or $2k+1$ for some $k \in \mathbb{N}$,

Then there exists a $\mathbb{C}$-uniruled subvariety $\Xi_\mho \subset M$ of complex dimension at least $n-k$.

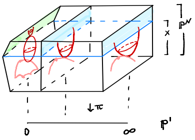


Figure: Degeneration to the normal cone

Main Application

Let M be a smooth affine variety of complex dimension n.

$$\cdots \to SH^{m-1}(M) \to SH_+^{m-1}(M) \xrightarrow{\delta} H^m(M) \to SH^m(M) \to \cdots$$

Definition (C.)

$\ell(M) := \min\{\deg([\alpha]) : [\alpha] \in H^*(M) \text{ with } 0 \neq [\alpha] \in \text{Image } \delta\}$.

$\ell(M)$ measures the co-dimension of maximal $\mathbb{C}$-uniruled subvariety.

Corollary (C.)

If $\ell(M) = 2k$ or $2k+1$ $(0 \leq k < n)$, then
M admits a $(n-k)$-dimensional family of affine lines.
Moreover, M admits a uniruled subvariety of dimension $n-k$.

Corollaries

Corollary (C.)

If $\ell(M) = 0$ or 1, then M is $\mathbb{C}$-uniruled.

Corollary

If $SH^*(M) = 0$, then $\ell(M) = 0$. Therefore M is -uniruled. (See also Theorem 5.4 in [Zhou19])

Proposition (C.)

$\ell(M \#_e N) = min\{\ell(M), \ell(N)\}$.

Proposition (C.)

$\ell(M \times N) = min\{\ell(M), \ell(N)\}$.

Invariance of Symplectic Cohomology

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & 0 \arrow[d] & \\
 \arrow[r] & SH^{k-1}(M_{\Supset k}) \arrow[r] \arrow[d, "\cong"'] & SH_{+}^{k-1}(M_{\Supset k}) \arrow[r] \arrow[d] & H^{k}(M_{\Supset k}) \arrow[r] \arrow[d] & SH^{k}(M_{\Supset k}) \arrow[r] \arrow[d, "\cong"'] & {} \\
 \arrow[r] & SH^{k-1}(M) \arrow[r] \arrow[d] & SH_{+}^{k-1}(M) \arrow[r] \arrow[d] & H^{k}(M) \arrow[r] \arrow[d] & SH^{k}(M) \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}
$$

Application of the Corollary

Theorem (C.'21)

Let W be a Weinstein manifold of $\dim W = 2n$ with $\ell(W) = \infty$. Suppose that we have a Weinstein manifold $W_{\ni k}$, obtained by attaching Weinstein k-handles to W $(k < n)$ so that $\operatorname{rk} H^k(W_{\ni k}) > \operatorname{rk} H^k(W)$. Then $\ell(W_{\ni k}) = k$.

Hence, if $W_{\ni k}$ is symplectomorphic to an affine variety $M_{\ni k}$, then $M_{\ni k}$ admits a $\mathbb{C}$-uniruled subvariety of complex dimension $\lceil n - \frac{k}{2} \rceil$.

Lemma (Cieliebak '02)

Subcritical Weinstein handle attachment does not change symplectic cohomology: $SH^*(M_{\ni k}) \cong SH^*(M)$.

Cylindrical Affine Variety

An affine variety M is called **cylindrical**

if it contains a dense principal Zariski open subset
$U = M \setminus (f = 0) \cong \mathbb{C} \times M'$,

for some $f \in \mathcal{O}(X)$, for an affine variety M'
(Kishimoto, Prokhorov, Zaidenberg,'11)

By Künneth formula,
$\mathbf{SH}^*(\mathbf{M} \setminus (\mathbf{f} = \mathbf{0})) \cong \mathbf{SH}^*(\mathbb{C}) \times \mathbf{SH}^*(\mathbf{M}') = \mathbf{0}$.

Is $\mathbf{SH}^*(\mathbf{M}) = \mathbf{0}$?

Theorem (C.'22)

For a cylindrical affine variety M, $\widehat{SH}^*(M) \cong SH^*(\underline{M} \subset M) = 0$.

Key Idea

For a smooth hypersurface $Y \subset M$,

$\exists$ Spectral Sequence, $SH^*\big(\underline{M \setminus Y} \subset M \setminus Y\big) \Rightarrow SH^*(\underline{M} \subset M)$.

More precisely,

$SH^*_{D_1 \cup D_2}\big(K \setminus \mathbb{D}_2 \subset X \setminus (D_1 \cup D_2)\big) \Rightarrow SH^*_{D_1}(K \subset X \setminus D_1)$

Towards Log-Minimal Model Program of Affine 3-folds

Let M be a normal quasi-projective complex threefold and X be a normal projective threefold compactifying M with $D := X \setminus M$. Choose a point $q \in D$ where X is smooth. Let $f : \tilde{X} \to X$ be the **weighted blow-up at q with weights** $(1, 1, b)$, where $b \in$. Let $E \cong \mathbb{P}^2_{(1,1,b)}$ be the exceptional divisor of f and $\tilde{D} := f^{-1}(D)$.

Definition (Kishimoto, '06)

$\tilde{M} := \tilde{X} \setminus \tilde{D}$ is a **half-point attachment** to $M : X \setminus D$.

Theorem (C.'22 Spectral Sequence)

Let $\tilde{M}$ be a half-point attachment to M at a smooth point on a hypersurface: $\tilde{M} := \mathcal{B}l_p A \setminus \tilde{Y} = M \cup (E \cup \tilde{M})$ and $\tilde{M}, M$ smooth. Then, $E^1 : SH^*(M) \cong SH^*(\tilde{M} \setminus (E \cup \tilde{M})) \Rightarrow SH^*(\tilde{M})$. Moreover, $\ell(\tilde{M}) \leq \ell(M)$.

Remarks on Symplectic Cohomology

- (Viterbo '99) $SH^*(T^*L) \cong H_*(\mathcal{L}L)$ ($\mathcal{L}L$, the free loop space of L).
- (Abouzaid '10) $SH^*(M) \cong HH^*(\mathcal{W}Fuk(M))$.
- (Pascaleff '19) SH^* of log-CY surfaces.
- (Ekholm–Lekili '23) Weinstein manifold X with an exact Lagrangian submanifold L, with ideal contact boundary with Legendrian submanifold Λ. Chekanov-Eliashberg DG-algebra of the Legendrian and the Lagrangian Floer cohomology of the Lagrangian are Koszul dual. ("Generalization between $C_{-*}(\Omega L)$" (ΩL and $C^*(L)$, the based loop space of L).
- (Borman–Sheridan–Varolgunes '22) Quantum cohomology as a deformation of symplectic cohomology.
- (Abouzaid–Groman–Varolgunes '22) Framed E_2 operad acting on Hamiltonian Floer theory $\Rightarrow$ The relative symplectic cohomology group carries a natural BV-algebra structure.

Thank you!