Constructing cospectral hypergraphs

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Alan Turing Institute London
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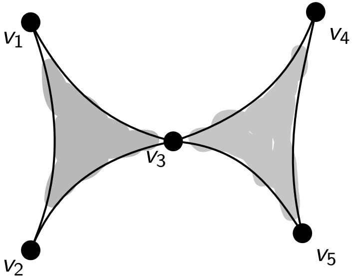
Outline

1. Introduction
2. Constructing cospectral hypergraphs with respect to tensors
3. Constructing cospectral hypergraphs with respect to matrices
4. Conclusion and future research

Introduction

Hypergraphs



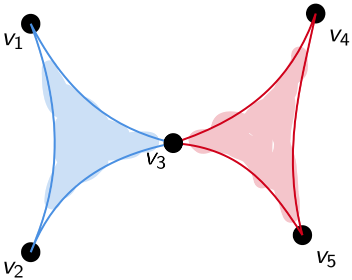
A **hypergraph** is a generalization of a graph in which an edge may have any number of vertices.

k-**uniform hypergraph**: every edge holds k vertices.

EXAMPLE. 3-uniform hypergraph on 5 vertices, the edge set is $E = \{v_1 v_2 v_3, v_3 v_4 v_5\}$.

EXAMPLE. 2-uniform hypergraph = (ordinary) graph.

Hypergraphs



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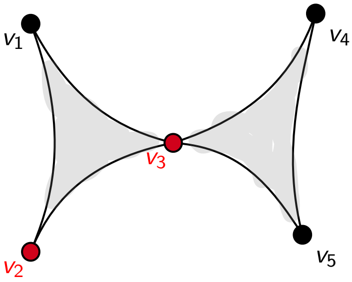
Hypergraphs



The **neighborhood** of a set U, $\Gamma(U)$, is a set of all vertices x such that $x \cup U$ is an edge.

EXAMPLE. The neighborhood of $\{v_3\}$ is $\Gamma(v_3) = \{v_1, v_2, v_4, v_5\}$.
EXAMPLE. The neighborhood of $\{v_2, v_3\}$ is $\Gamma(v_2, v_3) = \{v_1\}$.

Spectral graph theory

Spectral graph theory ($\sim$1950s) aims to obtain structural information about graphs from their spectra.

Many known results for *graphs* (Van Dam, Haemers, 2003):

- Regularity and bipartiteness can be determined from the spectrum
- Sharp eigenvalue bounds on NP-hard graph parameters (independence/chromatic number, . . .)
- Almost all trees and strongly regular graphs have cospectral mates
- Haemers's conjecture (2003): *Almost all graphs are determined by their spectrum*
- . . .

Spectral hypergraph theory

Spectral hypergraph theory ($\sim$1990s) aims to obtain structural information about hypergraphs from their spectra.

Some results in spectral hypergraph theory include (Cooper, Dutle, 2012):

- Regularity and bipartiteness can be determined from the hypergraph spectrum
- Eigenvalue bounds on the chromatic number of (oriented) hypergraphs (A., Mulas, Zhang, 2021)
- ...

Spectral hypergraph theory

Very few results are known for spectral characterizations of hypergraphs.

(Bu, Zhou, Wei, 2014) showed that the following families of hypergraphs are determined by their spectra:

- complete k-uniform hypergraphs and their complements,
- complete k-uniform hypergraphs without one edge,
- subhypergraphs of complete $(n-1)$-uniform hypergraphs.

OUR WISH: fill this literature gap.

Goal

Two hypergraphs are **cospectral** if they have the same spectrum (eigenvalues).

Studying *cospectral* graphs (hypergraphs) helps us reveal which structural properties cannot be deduced from the spectra.

Methods to construct cospectral *graphs* include:

- GM-switching (Godsil, McKay, 1982),
- *WQH-switching* (Wang, Qiu, Hu, 2019),
- ...

OUR GOAL is to obtain new methods to construct *cospectral hypergraphs*.

Representations of hypergraphs

The spectrum of a hypergraph can be calculated in different ways based on how the hypergraph is represented:

1. adjacency tensor, or hypermatrix (Cooper, Dutle, 2012),
2. integer matrix with number of common edges as its entries (Feng, Li, 1996),
3. ...

We focus on the *first two representations*.

Switching methods for hypergraphs

The GM-switching procedure has been extended to hypergraphs:

	GM-switching (Godsil, McKay, 1982) ↓	WQH-switching (Wang, Qiu, Hu, 2019) ↓
adjacency tensor:	(Bu, Zhou, Wei, 2014)	?
matrix representation:	(Sarkar, Banerjee, 2020)	?

Constructing cospectral hypergraphs with respect to tensors

Adjacency tensor of a hypergraph

For a *k-uniform hypergraph* on n vertices we can define the **adjacency tensor** $\mathcal{A} = (a_{i_1 \cdots i_k})$ of order k (length string of vertices) and dimension n (number of vertices):

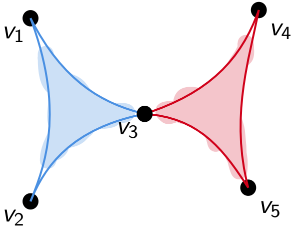
$$a_{i_1 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \{i_1, \ldots, i_k\} \in E, \\ 0, & \text{otherwise.} \end{cases}$$

REMARK 1. The adjacency tensor can be only defined for *uniform* hypergraphs.

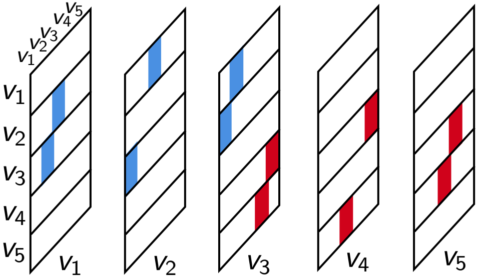
REMARK 2. Computing eigenvalues of a tensor is NP-hard (Hillar, Lim, 2013). Reduction is to square quadratic feasibility problem (is there an x such that $x^\top Ax = 0$), which in turn reduced to graph 3-colorability problem.

Adjacency tensor of a hypergraph

$$a_{i_1 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!} & \{i_1, \ldots, i_k\} \in E, \\ 0 & \text{otherwise.} \end{cases}$$



Example. Adjacency tensor of order 3 and dimension 5 corresponding to the 3-uniform hypergraph:



Eigenvalues of a tensor

What are the eigenvalues of a tensor?

For a MATRIX A of dimension n:

- λ is a root of the characteristic polynomial $\varphi_A(\lambda) = \det(\lambda I_n - A)$, or equivalently,
- λ is an *eigenvalue* if $Ax = \lambda x$ for some vector $x \neq 0$ and $x^\top x = 1$.

Eigenvalues of a tensor

What are the eigenvalues of a tensor?

For a TENSOR $\mathcal{A}$ of order k and dimension n there are *two* different definitions (Qi, 2005) and (Lim, 2005):

- λ is an **eigenvalue** if it is a root of the characteristic polynomial $\Phi_{\mathcal{A}}(\lambda) = \det(\lambda \mathcal{I}_n - \mathcal{A})$.
- λ is an **E-eigenvalue** if $\mathcal{A}x = \lambda x$ for some $x \neq 0$ and $x^\top x = 1$. The tensor-vector product in $\mathcal{A}x$ is defined similarly to the usual matrix-vector product (Shao, 2013).

Two hypergraphs are **cospectral (E-cospectral)** if they have the same eigenvalues (E-eigenvalues).

In this talk we will construct hypergraphs that are E-cospectral.

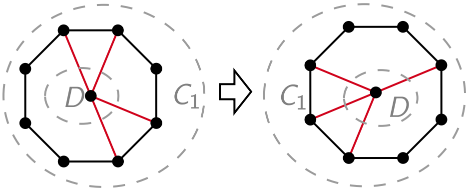
GM-switching (ordinary graphs)

Theorem (Godsil, McKay, 1982)

Let G be a graph whose vertex set admits a partition $C_1 \cup C_2 \cup \cdots \cup C_m \cup D$ such that:

1. For any $i \leq m$ each vertex in D has either 0, $\frac{1}{2}|C_i|$, or $|C_i|$ neighbors in C_i.
2. Equitable partition: for all $i, j \leq m$ every vertex in C_i has the same number of neighbors in C_j.

To construct the graph H, for any $v \in D$ that has $\frac{1}{2}|C_i|$ neighbors in C_i switch the adjacency of $\{u, v\}$ for all $u \in C_i$. Then H is a cospectral graph with G.

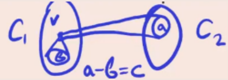


WQH-switching (ordinary graphs)

Theorem (Wang, Qiu, Hu, 2019)

Let G be a graph whose vertex set admits a partition $C_1 \cup C_2 \cup D$ such that:

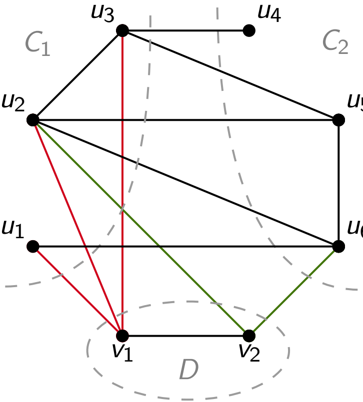
1. $|C_1| = |C_2|$.
2. There exists a constant c such that for any $v \in C_i$ we have $|\Gamma(v) \cap C_j| - |\Gamma(v) \cap C_i| = c$, where $\{i, j\} = \{1, 2\}$.



3. For any vertex $v \in D$ we have either $\Gamma(v) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$ or $|\Gamma(v) \cap C_1| = |\Gamma(v) \cap C_2|$.

To construct the graph H, for any $v \in D$ such that $\Gamma(v) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$ switch the adjacency of $\{u, v\}$ for any $u \in C_1 \cup C_2$. Then H is a cospectral graph with G.

WQH-switching (example)



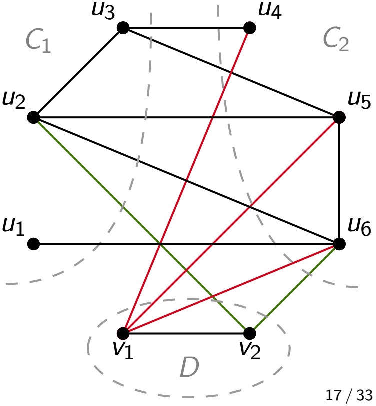
$\Gamma(v_1) \cap (C_1 \cup C_2) = C_1$

$|\Gamma(v_2) \cap C_1| = |\Gamma(v_2) \cap C_2|$

$|\Gamma(u_2) \cap C_2| - |\Gamma(u_2) \cap C_1| = 2 - 1 = 1$

$|\Gamma(u_1) \cap C_2| - |\Gamma(u_1) \cap C_1| = 1 - 0 = 1$

$\dots$



Switching edges: $v_1 u_1, v_1 u_2, v_1 u_3 \Rightarrow v_1 u_4, v_1 u_5, v_1 u_6$

WQH-switching (ordinary graphs)

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To construct a graph H, for any $v \in D$ such that $\Gamma(v) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$ switch the adjacency of $\{u, v\}$ for any $u \in C_1 \cup C_2$. Then H is a cospectral graph with G.

A generalized version of this switching for a partition of the vertices into $2m + 1$ subsets $C_1 \cup \cdots \cup C_{2m} \cup D$ was described by (Qiu, Ju, Wang, 2020).

New switching for hypergraphs (tensors)

Theorem 1 (A., Khramova, 2024)

Let G be a k-uniform hypergraph whose vertex set admits a partition $C_1 \cup C_2 \cup D$, and such that:

1. $|C_1| = |C_2|$.
2. Any edge has at least $k-1$ vertices in D.
3. For any $k-1$ distinct vertices $u_2, \ldots, u_k$ from D, we have $\Gamma(u_2, \ldots, u_k) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$ or $|\Gamma(u_2, \ldots, u_k) \cap C_1| = |\Gamma(u_2, \ldots, u_k) \cap C_2|$.

To construct the hypergraph H, for any $\{u_2, \ldots, u_k\} \subseteq D$ such that its neighbors in $C_1 \cup C_2$ are all in C_1 (or C_2), switch the adjacency of $\{u_1, \ldots, u_k\}$ for all $u_1 \in C_1 \cup C_2$. Then H is a k-uniform E-cospectral hypergraph with G.

New switching for hypergraphs (tensors)

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Condition 2 implies that

in $C_1 \cup C_2$ no two vertices are adjacent (C_1, C_2 cocliques).

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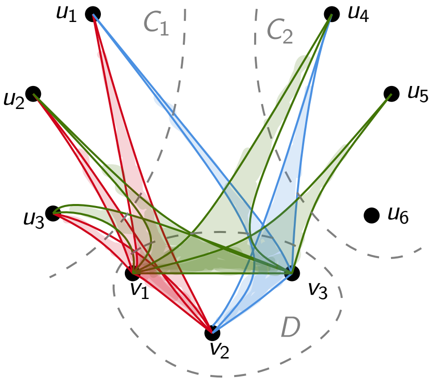
Condition 3 considers the neighborhood of $k-1$ vertices in D.

New switching for hypergraphs (tensors)

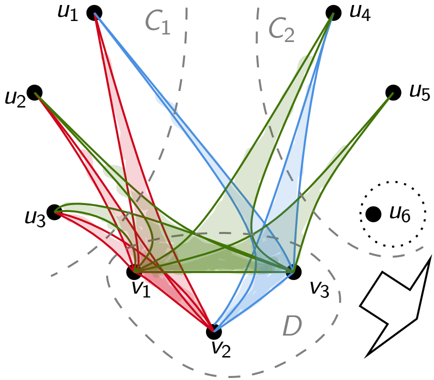


Edges:

		$v_1v_3u_2$
	$v_1v_2u_1$	$v_1v_3u_3$
$v_2v_3u_1$	$v_1v_2u_2$	$v_1v_3u_4$
$v_2v_3u_4$	$v_1v_2u_3$	$v_1v_3u_5$

- $|C_1| = |C_2| = 3$.
- Every edge has 2 vertices in D (this is a 3-uniform hypergraph).
- $\Gamma(v_1, v_2) = C_1$ (no neighbors in C_2);
- v_2, v_3 have one neighbor in each C_1 and C_2;
- v_1, v_3 have 2 neighbors in each C_1 and C_2.

New switching for hypergraphs (tensors)



Common edges:

$v_2 v_3 u_1$

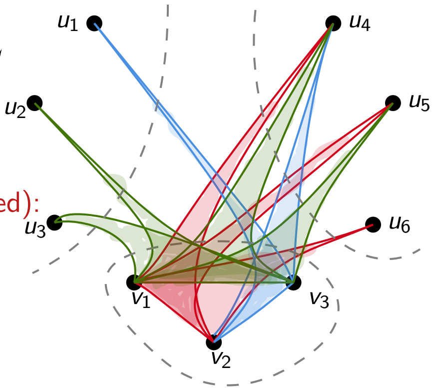
$v_2 v_3 u_4$

$v_1 v_3 u_2$

$v_1 v_3 u_3$

$v_1 v_3 u_4$

$v_1 v_3 u_5$



Switching edges (condition 3 red):

$v_1 v_2 u_1$	$v_1 v_2 u_4$
$v_1 v_2 u_2$	$v_1 v_2 u_5$
$v_1 v_2 u_3$	$v_1 v_2 u_6$

New switching for hypergraphs (tensors)

Theorem 1+ (A., Khramova, 2024)

Let G be a k-uniform hypergraph whose vertex set admits a partition $C_1 \cup C_2 \cup \cdots \cup C_{2m} \cup D$ for some $m \geq 1$, and such that:

1. $|C_i| = |C_{i+1}|$ for all odd $i < 2m$.
2. Any edge has at least $k-1$ vertices in D.
3. For any odd $i < 2m$ and $k-1$ distinct vertices $u_2, \ldots, u_k$ from D, we have $\Gamma(u_2, \ldots, u_k) \cap (C_i \cup C_{i+1}) \in \{C_i, C_{i+1}\}$ or $|\Gamma(u_2, \ldots, u_k) \cap C_i| = |\Gamma(u_2, \ldots, u_k) \cap C_{i+1}|$.

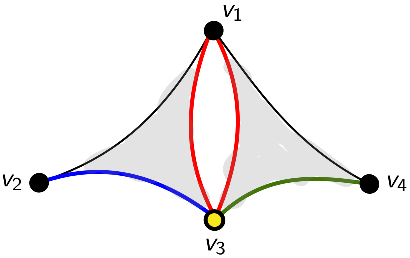
To construct the hypergraph H, for any $\{u_2, \ldots, u_k\} \subseteq D$ such that its neighbors in $C_i \cup C_{i+1}$ are all in C_i (or C_{i+1}), switch the adjacency of $\{u_1, \ldots, u_k\}$ for all $u_1 \in C_i \cup C_{i+1}$. Then H is a k-uniform E-cospectral ypergraph with G.

Constructing cospectral hypergraphs with respect to matrices

Matrix representation of a hypergraph

First proposed by (Feng, Li, 1996), and a similar definition is used by (Sarkar, Banerjee, 2020) to define a GM-switching for hypergraphs.

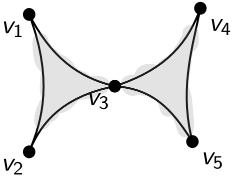
$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$



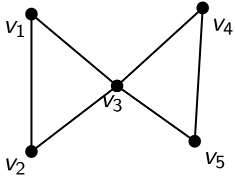
The entry a_{ij} of $A = (a_{ij})$ is the number of edges that contain both v_i and v_j.

Matrix representation of a hypergraph: downside

The matrix can be interpreted as an adjacency matrix of a multigraph; so with no additional information the hypergraph is not uniquely determined.



$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$



It is not as immediately clear what the edges are just from the matrix alone as it is when using the tensor definition, but the computation of eigenvalues can be done in polynomial time.

Hypergraph representations comparison

Tensor

+ contains full information about the hypergraph
+ the edge list is trivially obtained
+ existing spectral characterization results
- only applies to uniform hypergraphs
- computing eigenvalues is NP-hard

Matrix

- not enough to uniquely determine a hypergraph
- obtaining the edge list requires extra calculation
+ can be defined for non-uniform hypergraphs
+ computing eigenvalues can be done in polynomial time
+ more feasible for use in related fields and applications (random hypergraphs, neural networks, . . .)

New switching for hypergraphs (matrices)

Theorem 2 (A., Khramova, 2024)

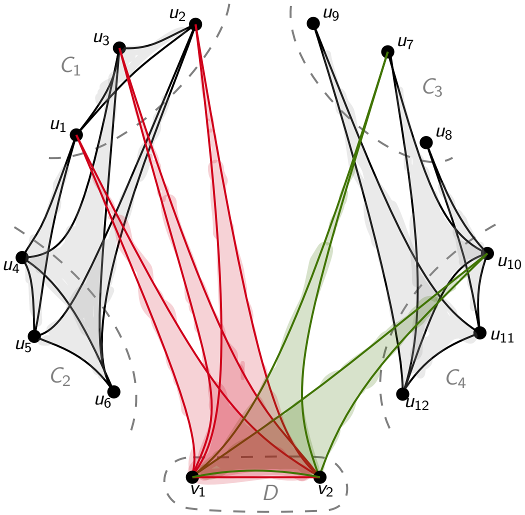
Let G be a k-uniform hypergraph whose vertex set admits a partition $C_1 \cup C_2 \cup \cdots \cup C_{2m} \cup D$ for some $m \geq 1$, and such that:

1. $|C_i| = t$ for all i and some t, while $|D| = k - 1$.
2. Any edge of G has 0 or $k - 1$ vertices in D.
3. For any odd $i < 2m$, we have either $\Gamma(D) \cap (C_i \cup C_{i+1}) = C_i$ or $|\Gamma(D) \cap C_i| = |\Gamma(D) \cap C_{i+1}|$.
4. For the adjacency matrix A and each $i, j \leq 2m$ there exists α_{ij} such that

$$\sum_{u \in C_i} A_{uv} = \sum_{u \in C_i} A_{vu} = \sum_{u \in C_j} A_{uw} = \sum_{u \in C_j} A_{wu} = \alpha_{ij} \text{ for } v \in C_j,\ w \in C_i.$$

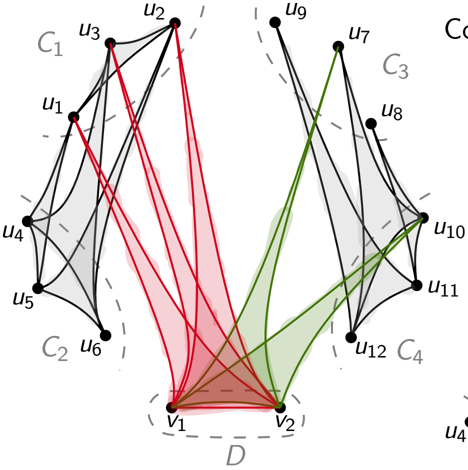
To construct the hypergraph H, remove all edges (v, D) such that $v \in C_i$ and $\Gamma(D) \supseteq C_i$ and add edges (u, D) with $u \in C_{i+1}$, for all odd $i < 2m$. Then H is cospectral to G with respect to matrix representation.

New switching for hypergraphs (matrices)



- $|C_i| = 3$ for $i = 1, 2, 3, 4$.
- $|D| = 2$ in a 3-uniform hypergraph.
- Every edge has 0 or 2 vertices in D.
- v_1, v_2 are adjacent to all of C_1 and none of C_2.
- v_1, v_2 are adjacent to one vertex in both C_3 or C_4.
- number of neighbors in C_j is the same for all $v \in C_i$, $i, j = 1, 2, 3, 4$ (equitable partition).

New switching for hypergraphs (matrices)



Common edges:

$u_1 u_2 u_3$
$u_1 u_4 u_5$
$u_2 u_5 u_6$
$u_3 u_4 u_6$

$u_7 u_{10} u_{12}$
$u_8 u_{10} u_{11}$
$u_9 u_{11} u_{12}$

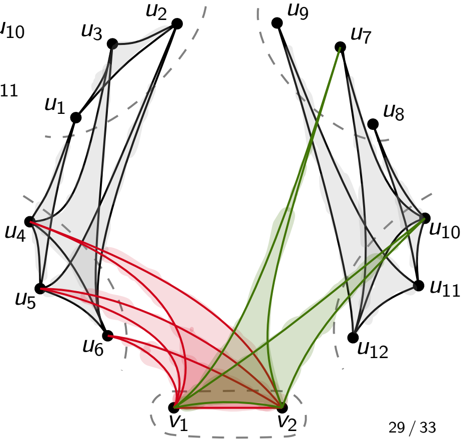
$u_7 v_1 v_2$
$u_{10} v_1 v_2$

Switching edges:

$v_1 v_2 u_1$
$v_1 v_2 u_2$
$v_1 v_2 u_3$

⇨

$v_1 v_2 u_4$
$v_1 v_2 u_5$
$v_1 v_2 u_6$



Conclusion and future research

Conclusion and future research

	GM-switching (Godsil, McKay, 1982) ↓	WQH-switching (Wang, Qiu, Hu, 2019) ↓	Other methods? ↓
tensor:	(Bu, Zhou, Wei, 2014)	**Theorem 1**	?
matrix:	(Sarkar, Banerjee, 2020)	**Theorem 2**	?

- What other results of spectral graph theory admit an extension to hypergraphs?
- Tools that could be used to derive new results on spectral characterizations of hypergraphs?
- Develop switching methods to construct cospectral oriented hypergraphs.
- Interpretation/application of cospectral hypergraphs in chemistry?

Thank you for listening!

For further details see:

A. Abiad, A.P. Khramova,
Constructing cospectral hypergraphs
Linear and Multilinear Algebra, to appear.

Proof idea (tensors)

Two graphs G and H with adjacency matrices $A(G)$ and $A(H)$ are cospectral if there exists a rational orthogonal matrix Q such that

$$A(H) = Q^\top A(G) Q.$$

In the case of **ordinary graphs**, such orthogonal matrix Q can be obtained for both GM-switching and WQH-switching.

Proof idea (tensors)

For **hypergraphs**, a similar observation is true:

Two hypergraphs G and H with adjacency tensors $\mathcal{A}(G)$ and $\mathcal{A}(H)$ are cospectral if there exists a rational orthogonal matrix Q such that

$$\mathcal{A}(H) = Q^\top \mathcal{A}(G) Q.$$

The challenge in the hypergraph context is to combinatorially define a switching operation that corresponds to an appropriate Q.