Linear SPDEs [a]

Salah Mohammed [b]

http://sfde.math.siu.edu/

Isaac Newton Institute

Cambridge, UK

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[a] [M-Z]

[b] Department of Mathematics, SIU-C, Carbondale, Illinois, USA

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Does the following linear stochastic evolution equation (see):

$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + Bv(t) \circ dW(t),\ t > 0, \\ v(0) &= Y \end{aligned}\right\} \qquad (1)$$

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Answer:

YES!

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Answer:

YES! (provided Y is sufficiently regular).

Strategy

- Replace Y in see (1) by a deterministic initial condition x in H and get the corresponding (equivalent) Itô see:

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + B^0u(t,x)\,dt \\ &\qquad + Bu(t,x)\,dW(t),\ t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \quad (2)$$

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  with $B^0u(t,x)\,dt$ the Stratonovich correction.

- View the mild solution of the see (2) as a function (cocycle) $U(t,x,\omega)$ of three variables (t,x,ω): Itô differentiable in t, continuous linear in x and Malliavin smooth in ω.

Strategy-Contd

- Consider the Stratonovich version of the Itô see (2):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + Bu(t,x)\circ dW(t),\ t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \qquad (2')$$

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- *In the above linear see, is it justified to replace the deterministic initial condition* x *by an arbitrary random variable* Y (substitution theorem)?

Strategy-Contd

- If Yes, then get back the anticipating Stratonovich see (1) again:

$$\left.\begin{aligned} dU(t,Y) = -AU(t,Y)\,dt + BU(t,Y) \circ dW(t), \\ t > 0 \\ U(0,Y) = Y \end{aligned}\right\} \quad (1)$$

by taking $v(t) := U(t,Y),\ t \geq 0.$

Difficulties

- Affirmative answer for the above question is known for a wide class of finite-dimensional sde's via substitution theorems ([Nu.1-2]).

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- Known substitution theorems require a level of regularity of the cocycle $U(t, x, \omega)$ in t that is inconsistent with infinite-dimensionality of the stochastic dynamics (Cf. Theorem 3.2.6 [Nu.1], Theorem 5.3.4 [Nu.2]).

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- Known substitution theorems require a level of regularity of the cocycle $U(t, x, \omega)$ in t that is inconsistent with infinite-dimensionality of the stochastic dynamics (Cf. Theorem 3.2.6 [Nu.1], Theorem 5.3.4 [Nu.2]).
- Existing substitution theorems work under restrictive finite-dimensional or compactness constraints ([G-Nu-S]).

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- Failure of Sobolev inequalities in infinite dimensions.

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- Develop global estimates on the semiflow generated by the spde.
- Use ideas and techniques of the Malliavin calculus: Assume Malliavin regularity of the initial condition -rather than imposing finite-dimensional or compactness restrictions on the values of the initial random condition.
- Use of Malliavin calculus techniques is necessary because the initial condition and the underlying stochastic dynamics are infinite-dimensional.

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Expect techniques developed in this analysis to yield similar substitution theorems for semiflows induced by quasilinear spde's (e.g. stochastic Burgers and Navier-Stokes equations).

Global moment estimates on the cocycle are interesting in their own right–also relevant for analysis of the Malliavin covariance of the anticipating random orthogonal projections on the invariant subspaces.

The Set-up

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- Wiener shifts $\theta : \mathbf{R} \times \Omega \to \Omega$: Group of P-preserving ergodic transformations on $(\Omega, \mathcal{F}, P)$:

$$\theta(t, \omega)(s) := \omega(t + s) - \omega(t), \quad t, s \in \mathbf{R}, \omega \in \Omega.$$

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- $H :=$ real (separable) Hilbert space, norm $|\cdot|_H$.
- $\mathcal{B}(H) :=$ Borel σ-algebra of H.
- $L(H) :=$ Banach space of all bounded linear operators $H \to H$ given the uniform operator norm $\|\cdot\|_{L(H)}$.

Set-up: Brownian Motion

- $W :=$ E-valued Brownian motion $W : \mathbf{R} \times \Omega \to E$ with separable covariance Hilbert space $K \subset E$, Hilbert-Schmidt embedding.

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  $\{f_k : k \geq 1\}$:= complete orthonormal basis of K; $W^k, k \geq 1$, standard independent one-dimensional Wiener processes ([D-Z.1], Chapter 4).

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- (W, θ) is a *helix*:
  $W(t_1 + t_2, \omega) - W(t_1, \omega) = W(t_2, \theta(t_1, \omega))$

Set-up-contd

- $L_2(K, H) :=$ Hilbert space of all Hilbert-Schmidt operators $S : K \to H$, with norm

$$\|S\|_2 := \left[\sum_{k=1}^{\infty} |S(f_k)|_H^2\right]^{1/2}$$

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- Suppose $B : H \longrightarrow L_2(K, H)$ is a bounded linear operator. Define $B_k \in L(H)$ by

$$B_k(x) := B(x)(f_k),\ x \in H,\ k \geq 1;$$

and assume $\sum\limits_{k=1}^{\infty} \|B_k\|^2$ converges.

Set-up: The linear SEE

Consider the linear Itô stochastic evolution equation (see):

$$\left.\begin{aligned} du(t,x) &= -Au(t,x)\,dt + \frac{1}{2}\sum_{k=1}^{\infty} B_k^2 u(t,x)\,dt \\ &\qquad + Bu(t,x)\,dW(t),\ t>0 \\ u(0,x) &= x \in H \end{aligned}\right\} \qquad (2)$$

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in H.

$A : D(A) \subset H \to H$ is a closed linear operator on H.

The Set-up-contd

Assume A has a complete orthonormal system of eigenvectors $\{e_n : n \geq 1\}$ with corresponding positive eigenvalues $\{\mu_n, n \geq 1\}$; i.e., $Ae_n = \mu_n e_n,\ n \geq 1.$

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The Itô stochastic integral in the see (2) is defined in the following sense ([D-Z.1], Chapter 4):

Set-up: The Itô Integral

Let $\psi : [0, a] \times \Omega \to L_2(K, H)$ be jointly measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted and

$$\int_0^a E\|\psi(t)\|^2_{L_2(K,H)}\, dt < \infty.$$

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Set

$$\int_0^a \psi(t)\, dW(t) := \sum_{k=1}^{\infty} \int_0^a \psi(t)(f_k)\, dW^k(t)$$

where the H-valued Itô integrals on the right hand side are with respect to the one-dimensional Wiener processes $W^k,\ k \geq 1$.

The Itô Integral-contd

Series converges in $L^2(\Omega, H)$ because

$$\sum_{k=1}^{\infty} E\left|\int_0^a \psi(t)(f_k)\,dW^k(t)\right|^2 = \int_0^a E\|\psi(t)\|^2_{L_2(K,H)}\,dt$$

$$< \infty.$$

Standing Hypotheses

- *Hypothesis* (A_1): $\displaystyle\sum_{n=1}^{\infty} \mu_n^{-1} \|B(e_n)\|^2_{L_2(K,H)} < \infty.$

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- *Hypothesis (B)*: $B : H \to L_2(K, H)$ extends to a bounded linear operator $B \in L(H, L(E, H))$ and $\sum_{k=1}^{\infty} \|B_k\|^2 < \infty$. Recall that $B_k \in L(H)$ are defined by

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  - (a) The operator $B : H \longrightarrow L_2(K, H)$ is Hilbert-Schmidt.
  - (b) $\liminf_{n\to\infty} \mu_n > 0$.
- Requirement (b) above is satisfied if $A = -\Delta$, where Δ is the Laplacian on a compact smooth d-dimensional Riemannian manifold M with boundary, under Dirichlet boundary conditions.
- No restriction on $\dim M$ under (A_1) for spdes.

Mild Solutions

A mild solution of the linear see (2) is a family of $(\mathcal{B}(\mathbf{R}^+) \otimes \mathcal{F}, \mathcal{B}(H))$-measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes $u(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H, \;\; x \in H$, satisfying the following stochastic integral equation:

$$u(t,x,\cdot) = T_t x + \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t T_{t-s}B_k^2 u(s,x,\cdot)\,ds + \int_0^t T_{t-s}Bu(s,x,\cdot)\,dW(s), \quad t \geq 0, \qquad (2')$$

([D-Z.1-2]).

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- $U(t_1 + t_2, \cdot, \omega) = U(t_2, \cdot, \theta(t_1, \omega)) \circ U(t_1, \cdot, \omega)$ for all $t_1, t_2 \in \mathbf{R}^+$, all $\omega \in \Omega$.

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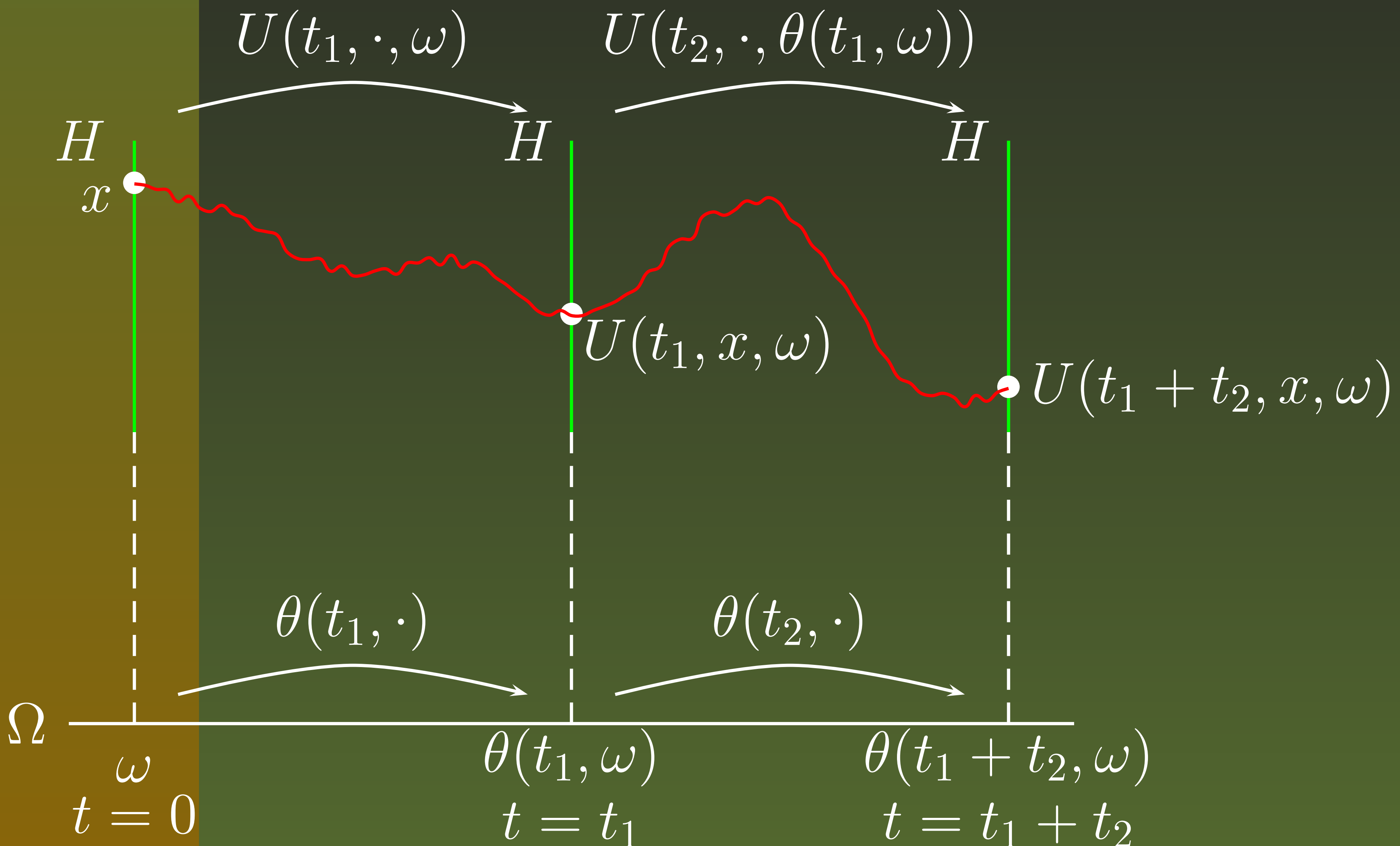
- $U(0, x, \omega) = x$ for all $x \in H, \omega \in \Omega$.

The Cocycle Property



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Theorem 1:

Under Hypotheses (B) and (A_1), the mild solutions of the see (2) admit a version $U : \mathbf{R}^+ \times \Omega \to L(H)$ *satisfying the following:*

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Under Hypotheses (B) and (A_1), the mild solutions of the see (2) admit a version $U : \mathbf{R}^+ \times \Omega \to L(H)$ satisfying the following:

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- $E \sup_{0\leq t_1, t_2 \leq a} \|U(t_1, \cdot, \theta(t_2, \omega))\|_{L(H)}^{2p} < \infty, \ p \geq 1.$

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- $E \sup_{0\leq t_1,t_2\leq a} \|U(t_1,\cdot,\theta(t_2,\omega))\|_{L(H)}^{2p} < \infty,\ p \geq 1.$
- *(U,θ) is a perfect linear cocycle:*

$$U(t_1+t_2,\cdot,\omega) = U(t_2,\cdot,\theta(t_1,\omega)) \circ U(t_1,\cdot,\omega)$$

for all $t_1, t_2 \geq 0$ and all $\omega \in \Omega$.

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- Use chaos-type expansion in $L_2(H)$.
- Prove convergence of the expansion in $L^{2p}(\Omega, L_2(H))$ via repeated application of moment estimates of the Itô integral.

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Now for the details:

Linear SEE

Existence of semiflows for mild solutions of the Itô linear see:

$$d\Phi(t,x,\cdot) = -A\Phi(t,x,\cdot)\,dt + B\Phi(t,x,\cdot)\,dW(t), \quad t > 0$$

$$u(0,x,\omega) = x \in H. \tag{2''}$$

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Recall that $A : D(A) \subset H \to H$ is a closed linear operator on the separable real Hilbert space H.

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e.g. $A = -\Delta$ on compact smooth Riemannian manifold with Dirichlet boundary conditions.

Mild Solutions: Revisited

Recall that a *mild solution* of the linear see $(2'')$ is a family of jointly measurable, $(\mathcal{F}_t)_{t\geq 0}$-adapted processes

$$\Phi(\cdot, x, \cdot) : \mathbf{R}^+ \times \Omega \to H, \;\; x \in H$$

such that

$$\Phi(t, x, \cdot) = T_t x + \int_0^t T_{t-s} B\Phi(s, x, \cdot)\, dW(s), \quad t \geq 0.$$

Integral equation holds x-almost surely, $x \in H$.

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Integral equation holds x-almost surely, $x \in H$.

Is $\Phi(t, x, \cdot)$ pathwise continuous linear in x?

Kolmogorov Fails!

Kolmogorov's continuity theorem fails for random fields $I : L^2([0,1],\mathbf{R}) \to L^2(\Omega,\mathbf{R})$

$$I(x) := \int_0^1 x(t)\,dW(t), \quad x \in L^2([0,1],\mathbf{R}).$$

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No continuous (or even measurable linear!) selection

$$L^2([0,1],\mathbf{R}) \times \Omega \to \mathbf{R}$$

$$(x,\omega) \mapsto I(x,\omega)$$

of I ([Mo.1], pp. 144-148).

Lifting

- Lift semigroup $T_t, t \geq 0$, to a strongly continuous semigroup of bounded linear operators
$\tilde{T}_t : L_2(K, H) \to L_2(K, H), t \geq 0$, via composition
$\tilde{T}_t(C) := T_t \circ C,\ C \in L_2(K, H), t \geq 0.$

Lifting

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- Lift stochastic integral

$$\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s), \;\; x \in H, \, t \geq 0,$$

to $L_2(H)$ for adapted square-integrable $v : \mathbf{R}^+ \times \Omega \to L_2(H)$. Denote lifting by

$$\int_0^t T_{t-s} B v(s)\, dW(s) \in L_2(H).$$

Lifting-contd

That is:

$$\left[\int_0^t T_{t-s} B v(s)\, dW(s)\right](x) =$$

$$\int_0^t \tilde{T}_{t-s}(\{[B \circ v(s)](x)\})\, dW(s)$$

for all $t \geq 0$, x-a.s..

Lift then Iterate: “Chaos"!

For each $t > 0$ *and almost all* $\omega \in \Omega$, $\Phi(t, \cdot, \omega) \in L_2(H)$ *has “chaos-type” representation*

$$\Phi(t, \cdot, \cdot) = T_t + \sum_{n=1}^{\infty} \int_0^t T_{t-s_1} B \int_0^{s_1} T_{s_1-s_2} B \cdots \int_0^{s_{n-1}} T_{s_{n-1}-s_n} B T_{s_n} \, dW(s_n) \cdots dW(s_2) \, dW(s_1).$$

Iterated Itô stochastic integrals are lifted integrals in $L_2(H)$, *and series converges absolutely in* $L_2(H)$.

Helix approximations

Helix approximations

Approximate the cylindrical Wiener process W in the Stratonovich equivalent of (2$''$) by smooth processes $W_n : \mathbf{R}^+ \times \Omega \to E$, $n \geq 1$, where

$$W_n(t,\omega) := n \int_{t-1/n}^{t} W(u,\omega)\, du - n \int_{-1/n}^{0} W(u,\omega)\, du,$$

for $t \geq 0$, $\omega \in \Omega$.

Helix approximations

Approximate the cylindrical Wiener process W in the Stratonovich equivalent of (2″) by smooth processes $W_n : \mathbf{R}^+ \times \Omega \to E$, $n \geq 1$, where

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for $t \geq 0$, $\omega \in \Omega$.

Each approximating process W_n is a helix:

$$W_n(t_2, \theta(t_1,\omega)) = W_n(t_2 + t_1, \omega) - W_n(t_1, \omega), t_1, t_2 \geq 0.$$

Helix approximations-contd

Prove the cocycle property for the corresponding approximating flows; then let $n \to \infty$ in $L_2(H)$ to get the perfect cocycle (Φ, θ) for the reduced linear see (2′′).

Helix approximations-contd

Prove the cocycle property for the corresponding approximating flows; then let $n \to \infty$ in $L_2(H)$ to get the perfect cocycle (Φ, θ) for the reduced linear see ($2''$).

Obtain the linear cocycle (U, θ) for the see (2) as the unique solution of the random linear integral equation:

$$U(t,x,\omega) = \Phi(t,x,\omega) + \frac{1}{2}\sum_{k=1}^{\infty}\int_0^t \Phi(t-s, B_k^2 U(s,x,\omega), \theta(s,\omega))\, ds$$

for $t \geq 0$, $\omega \in \Omega$. □

Malliavin Regularity

Malliavin Regularity

Get estimates on Malliavin derivatives $\mathcal{D}_u U(t, x, \cdot)$, $u, t \in [0, a]$ and $x \in H$ of the linear cocycle $U : \mathbf{R}^+ \times H \times \Omega \to H$.

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Derivations are based on results in [M.Z.Z], Gronwall's lemma and the fact that W has stationary independent increments.

For any integer $p \geq 2$, denote by $\mathbb{D}^{1,p}(\Omega, H)$ the Sobolev space of all $\mathcal{F}$-measurable random variables $Y : \Omega \to H$ which are p-integrable together with their Malliavin derivatives $\mathcal{D}Y$ ([Nu.1-2]).

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We now state the main substitution theorem in this talk.

Substitution

Theorem 2: (The Substitution Theorem)

Assume Hypotheses (B) and (A_1). Let $U : \mathbf{R}^+ \times H \times \Omega \to H$ be the linear cocycle generated by the see (2). Let $Y \in \mathbb{D}^{1,4}(\Omega, H)$ be a random variable. Then $v(t) := U(t, Y)$, $t \geq 0$, is a mild solution of the (anticipating) Stratonovich see

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$$\left.\begin{aligned} dv(t) &= -Av(t)\,dt + Bv(t) \circ dW(t),\ t > 0, \\ v(0) &= Y \end{aligned}\right\} \qquad (1)$$

Outline of Proof-Contd

- Prove the substitution theorem when Y is replaced by its finite-dimensional projections Y_n: Use finite-dimensional projections to smooth out the semigroup T_t in t, and apply finite-dimensional substitution techniques.

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Outline of Proof-Contd

- Prove the substitution theorem when Y is replaced by its finite-dimensional projections Y_n: Use finite-dimensional projections to smooth out the semigroup T_t in t, and apply finite-dimensional substitution techniques.
- Rewrite each finite-dimensional anticipating Stratonovich integral in terms of a Skorohod integral plus a Lebesgue integral correction term.
- Take n to ∞ via the moment estimates on the cocycle, its Malliavin derivatives and Dominated Convergence.

More Estimates

Theorem 3:

In the see (2), assume Hypotheses (B) and (A_1).

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(i) *Let* $u, t \in [0, a]$. *Define*

$$V(t, \cdot) := U(t, \cdot) - T_t, \quad t \in [0, a].$$

Then $V(t, \cdot) \in \mathbb{D}^{1,2p}(\Omega, L_2(H))$ *and*

$$E\left[\sup_{u \le t \le a} \|\mathcal{D}_u V(t, \cdot)\|_{L_2(H)}^{2p}\right] < \infty$$

for all $p \geq 1$.

More Estimates-contd

More Estimates-contd

(ii)

$$E\left[\sup_{\substack{0<u,t\leq a\\ x\in H\setminus\{0\}}}\frac{|\mathcal{D}_u U(t,x,\cdot)|_H^{2p}}{|x|_H^{2p}}\right]<\infty$$

for all $p\geq 1$.

Finite-dimensional Projections

Objective:

To prove the substitution theorem when the random variable $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is replaced by its finite-dimensional projections on H.

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To prove the substitution theorem when the random variable $Y \in \mathbb{D}^{1,4}(\Omega, H)$ is replaced by its finite-dimensional projections on H.

$\{e_n : n \geq 1\} :=$ complete orthonormal system of eigenvectors of A.

$H_n := L\{e_i : 1 \leq i \leq n\}$, the n-dimensional linear subspace of H spanned by $\{e_i : 1 \leq i \leq n\}$, for each $n \geq 1$.

Projections-contd

Define the projections $P_n : H \to H_n$, $n \geq 1$, by

$$P_n(x) := \sum_{k=1}^{n} < x, e_k, > e_k, \qquad x \in H.$$

Projections-contd

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$$P_n(x) := \sum_{k=1}^{n} < x, e_k, > e_k, \quad x \in H.$$

Define $Y_n : \Omega \to H_n$ by

$$Y_n := P_n \circ Y, \quad n \geq 1.$$

Then $Y_n \to Y$ as $n \to \infty$ a.s.

Finite-dimensional Substitution

Theorem 4:

Assume (B) and (A_1) *and suppose* $Y \in \mathbb{D}^{1,4}(\Omega, H)$. *Then*

$$\left.\begin{aligned} dU(t,Y_n) &= -AU(t,Y_n)\,dt + BU(t,Y_n)\circ dW(t), t > 0, \\ U(0,Y_n) &= Y_n. \end{aligned}\right\} \tag{3}$$

for each $n \geq 1$.

Proof of Theorem 4

Proof still requires Malliavin calculus techniques, largely due to the underlying strongly continuous semi-group dynamics in $\{T_t\}_{t\geq 0}$.

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Rewrite see (2) in mild Stratonovich form:

$$U(t,x) = T_t(x) + \int_0^t T_{t-s} BU(s,x) \circ dW(s), \quad t > 0. \tag{2'''}$$

Proof of Theorem 4-contd

Sufficient to show that x in (2‴) can be replaced by Y_n:

$$\left.\begin{array}{c} U(t, Y_n) = T_t(Y_n) + \displaystyle\int_0^t T_{t-s} BU(s, Y_n) \circ dW(s), \\ t > 0,\ n \geq 1. \end{array}\right\} \tag{4}$$

Proof of Theorem 4-contd

To prove (4), we show that the random field

$$\int_0^t T_{t-s} BU(s,x) \circ dW(s), \quad x \in H_n,$$

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$$\int_0^t T_{t-s} BU(s,x) \circ dW(s) \Big|_{x=Y_n} = \int_0^t T_{t-s} BU(s,Y_n) \circ dW(s) \quad (5)$$

a.s. for fixed $t > 0$.

Proof of Theorem 4-contd

To prove (5), fix $m \geq 1$: H_m is invariant under T_t. Therefore, $T_{t-s}P_m$ is smooth in s. Hence by finite-dimensional substitutions ([Nu.1-2]):

$$\int_0^t T_{t-s}P_m BU(s,x) \circ dW(s) \Big|_{x=Y_n} = \int_0^t T_{t-s}P_m BU(s,Y_n) \circ dW(s) \tag{6}$$

a.s. for all $m, n \geq 1$.

Proof of Theorem 4-contd

Use global estimates on U to represent the Stratonovich integrals (in (5) and (6)) in terms of Skorohod integrals. Then pass to the limit as $m \to \infty$ in (6), using finite-dimensional substitutions, global estimates on U and dominated convergence. $\square$

Proof of Substitution Theorem 2

Step 1:

Suppose $Y \in \mathbb{D}^{1,4}(\Omega, H)$, and assume Hypothesis (B) and (A_1).

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a.s. for $t \geq 0$.

Proof of Theorem 2-contd

Step 2:

Pass to the limit as $n \to \infty$ in the finite-dimensional result:

$$\left.\begin{aligned} U(t, Y_n) = T_t(Y_n) + \int_0^t T_{t-s} BU(s, Y_n) \circ dW(s), \\ t > 0,\ n \geq 1. \end{aligned}\right\} \quad (8)$$

Localization

Denote by $\mathbb{L}^{1,2}$ the class of all processes $v : [0,t] \times \Omega \to H$ such that $v \in L^2([0,t] \times \Omega, H)$, $v(s,\cdot) \in \mathbb{D}^{1,2}(\Omega, H)$ for almost all $s \in [0,t]$ and $E[\int_0^t \int_0^t |\mathcal{D}_u v(s,\cdot)|_H^2 \, du \, ds] < \infty$.

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We say that v belongs to $\mathbb{L}^{1,2}_{loc}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:

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We say that v belongs to $\mathbb{L}^{1,2}_{loc}$ if there exists a sequence $(\Omega_m, v^m) \in \mathcal{F} \times \mathbb{L}^{1,2}$ with the following properties:

(i) $\Omega_m \uparrow \Omega$ as $m \to \infty$,

(ii) $v = v^m$ on Ω_m.

Proof of Theorem 2

Step 3:

The Stratonovich integral

$$\int_0^t T_{t-s} BU(s, Y) \circ dW(s)$$

in (7) is well-defined:

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The Stratonovich integral

$$\int_0^t T_{t-s} BU(s, Y) \circ dW(s)$$

in (7) is well-defined:
Sufficient to show that the process

$$v(s) := T_{t-s} BU(s, Y), s \leq t$$

is in $\mathbb{L}^{1,2}_{loc}$ ([Nu.2], Theorem 5.2.3).

Proof of Theorem 2

Localize v*:*

Proof of Theorem 2

Localize v:

$m \geq 1$ any integer. $\phi_m \in C_b^2(\mathbf{R}, \mathbf{R})$ a bump function such that $\phi_m(z) = 1$ for $|z| \leq m$ and $\phi_m(z) = 0$ for $|z| > m + 1$. Define

$$v^m(s) := v(s)\phi_m(|Y|_H), \; s \leq t.$$

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Then $v = v^m$ on $\Omega_m = \{\omega : |Y(\omega)|_H \leq m\}$ for each $m \geq 1$.

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$v^m \in \mathbb{L}^{1,2}$ for every $m \geq 1$ because $Y \in \mathbb{D}^{1,4}(\Omega, H)$ and the global moment estimates on U and its Malliavin derivatives.

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Then $v = v^m$ on $\Omega_m = \{\omega : |Y(\omega)|_H \leq m\}$ for each $m \geq 1$.
$v^m \in \mathbb{L}^{1,2}$ for every $m \geq 1$ because $Y \in \mathbb{D}^{1,4}(\Omega, H)$ and the global moment estimates on U and its Malliavin derivatives.
Hence v is Stratonovich integrable.

Easy Limits

Step 4:

Pass to the limit a.s. as $n \to \infty$ in (8). Get easy a.s. limits:

$$\lim_{n\to\infty} U(t, Y_n) = U(t, Y)$$

$$\lim_{n\to\infty} T_t(Y_n) = T_t(Y)$$

for each $t \geq 0$.

A Not-So-Easy Limit

Step 5:

A Not-So-Easy Limit

Step 5:

But following limit is non-trivial:

$$\left.\begin{aligned} &\lim_{n\to\infty}\int_0^t T_{t-s}BU(s,Y_n)\circ dW(s) \\ &\qquad = \int_0^t T_{t-s}BU(s,Y)\circ dW(s) \end{aligned}\right\} \qquad (9)$$

in probability for each $t \geq 0$.

Proof of Theorem 2-contd

Step 6:

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To prove (9), use local property of the Stratonovich integral:

Proof of Theorem 2-contd

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$$\int_0^t T_{t-s} BU(s,Y_n) \circ dW(s)$$

$$= \int_0^t T_{t-s} BU(s, Y_n)\phi_m(|Y|_H) \circ dW(s),$$

on $\Omega_m := \{\omega : |Y(\omega)|_H \leq m\}$;

Proof of Theorem 2-contd

and

$$\int_0^t T_{t-s} BU(s,Y) \circ dW(s)$$

$$= \int_0^t T_{t-s} BU(s,Y) \phi_m(|Y|_H) \circ dW(s)$$

on Ω_m for any fixed integer $m \geq 1$.

Proof of Theorem 2-contd

Step 7:

(9) will follow from

$$\lim_{n\to\infty} \int_0^t T_{t-s} BU(s, Y_n)\phi_m(|Y|_H) \circ dW(s) = \int_0^t T_{t-s} BU(s, Y)\phi_m(|Y|_H) \circ dW(s) \tag{10}$$

in probability for each $m \geq 1$.

Proof of Theorem 2-contd

To prove (10), fix $m \geq 1$ and let

$$g_n(s) := T_{t-s} BU(s, Y_n) \phi_m(|Y|_H),$$

$$g(s) := T_{t-s} BU(s, Y) \phi_m(|Y|_H)$$

for all $s \in [0, t]$. Then

$$\lim_{n \to \infty} E \left[\int_0^T \| g_n(s) - g(s) \|^2_{L_2(K,H)} \, ds \right] = 0 \qquad (11)$$

$$\lim_{n \to \infty} E \left[\int_0^T \int_0^T \| \mathcal{D}_u g_n(s) - \mathcal{D}_u g(s) \|^2_{L_2(K,H)} \, du \, ds \right] = 0. \qquad (12)$$

Proof of Theorem 2-contd

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Compute:

$$(\mathcal{D}_+ g)_u := \lim_{s \to u+} \mathcal{D}_u g(s), \ (\mathcal{D}_- g)_u := \lim_{s \to u-} \mathcal{D}_u g(s)$$

$$(\nabla g)_u := (\mathcal{D}_+ g)_u + (\mathcal{D}_- g)_u$$

Proof of Theorem 2-contd

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and use path continuity to get

$$\lim_{n \to \infty} (\nabla g_n)_u = (\nabla g)_u, \quad a.s.$$

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and use path continuity to get

$$\lim_{n \to \infty} (\nabla g_n)_u = (\nabla g)_u, \quad a.s.$$

Also get convergence in probability of the Skorohod integrals:

$$\lim_{n \to \infty} \int_0^t g_n(s)\, dW(s) = \int_0^t g(s)\, dW(s), \quad t \geq 0.$$

Proof of Theorem 2-contd

Step 7:

Proof of substitution theorem will be complete if:

$$\int_0^t g_n(s) \circ dW(s) = \int_0^t g_n(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g_n)_s \, ds, \tag{13}$$

for $n \geq 1$; and

Proof of Theorem 2-contd

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for $n \geq 1$; and

$$\int_0^t g(s) \circ dW(s) = \int_0^t g(s) dW(s) + \frac{1}{2} \int_0^t (\nabla g)_s \, ds \tag{14}$$

a.s.. Skorohod integrals on RHS.

Proof of Theorem 2-contd

Prove (13) and (14) from first principles, using approximations by Riemann sums: Lengthy computation.

Proof of Theorem 2-contd

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Step 8:

Take $n \to \infty$ in RHS of (13). $\square$

The Invariant Subspaces

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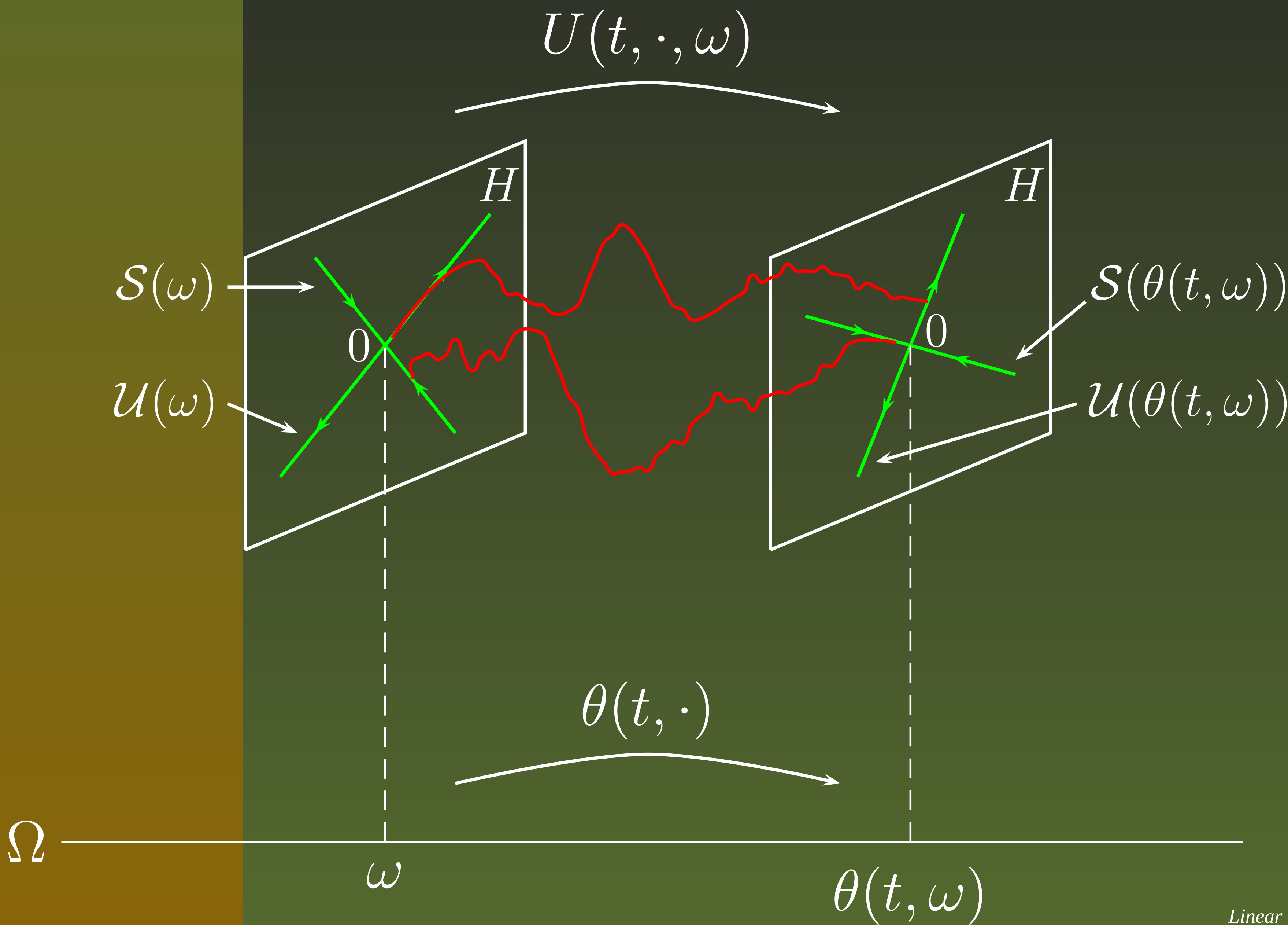
The restricted cocycle

$$U(t, \cdot, \omega)|\mathcal{U}(\omega) : \mathcal{U}(\omega) \to \mathcal{U}(\theta(t, \omega))$$

is a linear homeomorphism onto for each $t \geq 0, \omega \in \Omega$.

A Random Saddle



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Cf. work by Arnold and Imkeller (1995) and Imkeller (1997) on linear sodes.

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THE END!

THANK YOU!