Survey of the known algebraic solutions of Painlevé VI

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Classical example

Icosahedral rotation group A_5 of order 60

$$A_5 = \Delta_{235} = \langle a, b, c \mid a^2 = b^3 = c^5 = abc = 1 \rangle$$

- natural to look for ODEs on $\mathbb{P}^1 \setminus 3$ points with monodromy A_5

$$A_5 \subset SO_3(\mathbb{R}) \subset SO_3(\mathbb{C}) \cong PSL_2(\mathbb{C})$$

- look for connections on rank 2 vector bundles with projective monodromy A_5

Schwarz's List (1873)

Gauss hypergeometric equation

$\Rightarrow$ logarithmic connection $\left(\frac{A_1}{z} + \frac{A_2}{z-1}\right) dz$

- A_1, A_2 2×2 rank 1 matrices

  (twist by log. connection on line bundle)

- $A_1, A_2 \in sl_2(\mathbb{C})$ (2×2 trace free)

[connection on trivial principal $SL_2(\mathbb{C})$ bundle over $\mathbb{C}P^1$]

Algebraic horizontal sections classified by Schwarz

$\rightsquigarrow$ List with 15 entries:

1 dihedral family
2 Tetrahedral solutions
2 Octahedral solutions
10 Icosahedral solutions

[Rigid]

régulier. D'ailleurs, si l'intégrale générale de l'une des équations $\mathcal{E}(\lambda, \mu, \nu)$, $\mathcal{E}(1-\lambda, 1-\mu, \nu)$, $\mathcal{E}(\lambda, 1-\mu, 1-\nu)$, $\mathcal{E}(1-\lambda, \mu, 1-\nu)$ est une fonction algébrique de x, il en est évidemment de même des trois autres (n° 35).

Soient $\lambda'\pi$, $\mu'\pi$, $\nu'\pi$ les angles de celui des quatre triangles PQR, PQ'R, QP'R, P'Q'R pour lequel la somme des angles est la plus petite, les nombres λ', μ', ν' étant rangés par ordre de grandeur décroissante. Pour que l'intégrale générale de $E(\alpha, \beta, \gamma)$ soit une fonction algébrique, il faut et il suffit que les nombres λ', μ', ν', qui se déduisent de α, β, γ comme il a été expliqué, figurent dans le tableau ci-dessous de Schwarz :

	λ'	μ'	ν'	
(I)	$\frac{1}{2}$	$\frac{1}{2}$	»	double pyramide
(II)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$	Tétraèdre
(III)	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	
(IV)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	Cube et octaèdre
(V)	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	
(VI)	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$	a b c
(VII)	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{1}{3}$	b b d
(VIII)	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	b c c
(IX)	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{5}$	a c d
(X)	$\frac{3}{5}$	$\frac{1}{3}$	$\frac{1}{5}$	b c d
(XI)	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	Icosaèdre et dodécaèdre d d d
(XII)	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	b b c
(XIII)	$\frac{4}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	c c c
(XIV)	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{3}$	a b d
(XV)	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	b c d

A_5 conjugacy classes

a	½-turn
b	⅓-
c	⅕-
$d = c^2$	⅖-

Naive generalisations

One more pole :- $\sum_1^3 \frac{A_i}{z-a_i} dz$ WLOG $a_1, a_2, a_3 = 0, t, 1$

A $A_i \in sl_2(\mathbb{C})$

B A_i 3×3 rank 1

[Both minimally non-rigid – 2d moduli spaces]

Qn Analogue of Schwarz's list for these?

– can now answer this "nonrigid Schwarz list"

– still linear

Example of problem B:

Full symmetry group – icosahedral reflection group (order 120)

$$H = \left\langle r_1, r_2, r_3 \;\middle|\; \begin{array}{l} r_1^2 = r_2^2 = r_3^2 = 1 \\ (r_1 r_2)^2 = (r_2 r_3)^3 = (r_3 r_1)^5 = 1 \end{array} \right\rangle$$

$$\subset O_3(\mathbb{R}) \subset GL_3(\mathbb{C})$$

- look for connections on rank 3 bundles / $\mathbb{P}^1 \setminus 4$ points with monodromy H (generated by 3 reflections)

- (essentially) solved around 1997 by Dubrovin – Mazzocco

  3 inequivalent triples of generating reflections

  1 ~ K. Saito's icosahedral Frobenius manifold

  1 involves 10 pages of 40 digit integers

$$-1226684412907984419281022032089194096771900\,t^{8}$$
$$-1114701349894370233505605371103641055314707\,t^{9}$$
$$+706698148832598485833137372995728746006888\,t^{10}$$
$$-230885597278675059768074093486733449982986\,t^{11}$$
$$+40110760213781966306595755424591426952408\,t^{12}$$
$$-2944406938738808019234484282441173992613\,t^{13}$$
$$+299099898102561946553118326231329 56\,t^{14})\;x\,(1+x)\,y^{15}$$
$$+3\,(-193453115241036892998064298665955843444349 33$$
$$+165880840018062517894524148661179410853072546\,t$$
$$-433975351186661527899190510419861031681577223\,t^{2}$$
$$+515516306674309051714096086492072331808918060\,t^{3}$$
$$-283562876761607595979024343783955270990852289\,t^{4}$$
$$+350897178706520371665288657820712422849187 34\,t^{5}$$
$$+332979289901271870498313044573879436875789 09\,t^{6}$$
$$-129177642448516648728276204725560828032268 56\,t^{7}$$
$$-266713623245328356955979252488258143292463\,t^{8}$$
$$+555900198844440351814987030522263162652334\,t^{9}$$
$$+344809125199575823496923125385565831315595\,t^{10}$$
$$-325689072459807008457121908075371991483716\,t^{11}$$
$$+117388439783020206894897144460070846332949\,t^{12}$$
$$-211236880726863685681701964967539374371 82\,t^{13}$$
$$+156916158874243476028223548009010008225 5\,t^{14})\;x\,y^{16}$$
$$+3\,(978329976048894803021943300608357029668935 7$$
$$-593211193479185436599306765219843840421694 30\,t$$
$$+141416477837529651726686264572772822193430055\,t^{2}$$
$$-177096809878289456793903796377476455257673500\,t^{3}$$
$$+127907586479651422318564410835908192786763365\,t^{4}$$
$$-543726583091396407334392960210480497267466 98\,t^{5}$$
$$+134883943759832591783862690310778265413236 79\,t^{6}$$

Dubrovin – Mazzocco
Preprint ~ 1997

Nonlinear analogue — Painlevé VI equation

Explicit form of simplest (abelian) Gauss-Manin systems are Gauss hypergeometric equations

[periods of families of elliptic curves — Gauss]

Explicit form of simplest non-abelian Gauss-Manin connection is the Painlevé VI equation

- "nonlinear" analogue of hgeom. equation
- solutions branch at $0, 1, \infty \in \mathbb{P}^1$ (still)

Main question Analogue of Schwarz's list for P_{VI}?

C

- still open
- will describe what is known + methods used

Other motivations

- Often geometrically significant
  e.g. link with: Frobenius manifolds
  Poncelet problem / modular curves
  Elliptic fibrations

- Method to construct non-rigid Fuchsian systems with known monodromy (Riemann-Hilbert problem)

- P_{VI} is reduction of 4d ASDYM equations
  - so any solution should be interesting

- Good 'testing ground' for general Painlevé VI machinery

What is Painlevé VI ?

- explicit form of simplest non-Abelian Gauss–Manin connection
- equation controlling isomonodromy deformations of certain log connections/Fuchsian systems on $\mathbb{P}^1$
- most general 2nd order ODE with Painlevé property
- certain dimensional reduction of ASDYM equations
- equation related to certain elliptic integrals with moving endpoints (R. Fuchs/Manin)

The Painlevé VI equation (P_{VI}):

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

where the constants $\alpha, \beta, \gamma, \delta$ are related to the parameters $(\theta_1, \theta_2, \theta_3, \theta_4)$ by:

$$\alpha = (\theta_4 - 1)^2/2, \qquad \beta = -\theta_1^2/2, \qquad \gamma = \theta_3^2/2, \qquad \delta = (1 - \theta_2^2)/2.$$

Main properties (well-known)

- Painlevé property – critical singularities at $0, 1, \infty$:
  "Any local solution $y(t)$ near $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ extends to meromorphic function on universal cover"

- Trichotomy
  Any solution is either
  - a 'new' transcendental function
  - a solution of a 1st order Riccatti eqn
  - an algebraic function

- Governs isomonodromic deformations of rank 2 log. connections $\sum_1^3 \frac{A_i}{z - a_i}$ on $\mathbb{P}^1$ (type A)
  Eigenvalues $A_i = \pm\theta_i/2$ $\quad (A_4 = -\sum_1^3 A_i)$

- $W_{aff}(F_4)$ symmetry group (standard action on $\mathbb{C}^4 \ni (\theta_1, \theta_2, \theta_3, \theta_4)$)

Definition

An algebraic solution to P_{VI} is an irreducible polynomial $F(y,t) \in \mathbb{C}[y,t]$ s.t the algebraic function $y(t)$ defined implicitly by

$$F(y(t),t) = 0$$

solves P_{VI} for some value of the parameters

Definition'

... is an (irreducible compact) algebraic curve Π and two rational functions $y, t : \Pi \to \mathbb{P}^1$ s.t

① t is a Belyi map (branch locus $\subset \{0,1,\infty\}$)

② $y(t)$ solves P_{VI} (for some parameters)

Def^n

Π is a minimal Painlevé curve (or an "efficient parameterisation") if

$$\Pi \underset{\text{birational}}{\cong} \{ F(y,t) = 0 \}$$

Principal Invariants

- degree $=$ degree of Belyi map $t : \Pi \to \mathbb{P}^1$ (if Π minimal)

  $= \deg_y(F) \quad =$ "no. of branches"

- genus $=$ genus(Π) (if minimal)

  $=$ genus of function field $\mathbb{C}(y, t)$

Basic examples of algebraic solutions to Painlevé VI (Hitchin, Dubrovin):

Three-branch tetrahedral solution:

$$y = \frac{(s-1)(s+2)}{s(s+1)}, \qquad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)}$$

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (2/3, 1/3, 1/3, 2/3)$$

Four-branch dihedral solution:

$$y = \frac{s^2(s+2)}{s^2+s+1}, \qquad t = \frac{s^3(s+2)}{2s+1}$$

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/2, 1/2)$$

Four-branch octahedral solution:

$$y = \frac{(s-1)^2}{s(s-2)}, \qquad t = \frac{(s+1)(s-1)^3}{s^3(s-2)}$$

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/4, 1/4, 1/4, 1/4)$$

Basic *families* of algebraic solutions to Painlevé VI:

Square root family:

$$y = \pm\sqrt{t}$$

$$\theta_2 = \theta_3 \quad \text{and} \quad \theta_1 + \theta_4 = 1$$

Three-branch tetrahedral family:

$$y = \frac{(s-1)(s+2)}{s(s+1)}, \qquad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)}$$

$$\theta_1/2 = \theta_2 = \theta_3, \theta_4 = \frac{2}{3}$$

Four-branch dihedral family:

$$y = \frac{s^2(s+2)}{s^2+s+1}, \qquad t = \frac{s^3(s+2)}{2s+1}$$

$$\theta_1 = \theta_2 = \theta_3, \theta_4 = 1/2$$

Four-branch octahedral family:

$$y = \frac{(s-1)^2}{s(s-2)}, \qquad t = \frac{(s+1)(s-1)^3}{s^3(s-2)}$$

$$\theta_1 = \theta_2 = \theta_3, \theta_4 = 1 - 3\theta_1$$

Okamoto's affine Weyl group action

If $y(t)$ solves P_{II} with parameters $(\theta_1, \theta_2, \theta_3, \theta_4)$

then $y(t)$ ———————— $(-\theta_1, \theta_2, \theta_3, \theta_4)$

& $y(t)$ ———————— $(\theta_1, -\theta_2, \theta_3, \theta_4)$

& $y(t)$ ———————— $(\theta_1, \theta_2, -\theta_3, \theta_4)$

& $y(t)$ ———————— $(\theta_1, \theta_2, \theta_3, 2-\theta_4)$

– reflections in hyperplanes $\theta_i = 0$ $(i=1,2,3)$, $\theta_4 = 1$

Thm If defined, $y + \phi/x$ solves P_{II} with params

$$(\theta_1 - \phi, \theta_2 - \phi, \theta_3 - \phi, \theta_4 - \phi) \quad \text{where } \phi = \textstyle\sum_1^4 \theta_i/2$$

$$\& \quad x = \frac{1}{2}\left(\frac{(t-1)y' - \theta_1}{y} + \frac{y' - 1 - \theta_2}{y-t} - \frac{ty' + \theta_3}{y-1}\right).$$

– reflection in hyperplane $\sum \theta_i = 0$

– these 5 transformations generate a group isom. to $W_{aff}(D_4)$

$\hat{D}_4 =$

– Can add in S_4 syms of $\hat{D}_4$ (R. Fuchs / Schlesinger)

& get sym. group $\cong W_{aff}(F_4)$

$W_{aff}(F_4)$ is an infinite group

$$\cong \underbrace{W(F_4)}_{\text{finite reflection group (order 1152)}} \ltimes \underbrace{\Lambda(F_4)}_{\text{translation subgroup} \cong \mathbb{Z}^4}$$

$$\Lambda(F_4) = \langle \varepsilon_i \pm \varepsilon_j \rangle_{\mathbb{Z}}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ standard ON basis of $\mathbb{C}^4$

(coords θ_i on $\mathbb{C}^4$ so $\sum \theta_i \varepsilon_i \in \mathbb{C}^4$)

$W_{aff}(F_4)$ generated by reflections in the five hyperplanes: $\theta_2 = \theta_3$, $\theta_3 = \theta_4$, $\theta_4 = 0$, $\theta_1 = \theta_2 + \theta_3 + \theta_4$

$\theta_1 + \theta_2 = 1$

Shape of table so far

[Riccatti / Rational solutions Watanabe, Mazzocco, Yuan-Li]

4 continuous families $g=0$ $d=$ 2 $\sqrt{z}$ (Picard/Okamoto 3)
3
4
4 } Hitchin / Dubrovin

1 discete family $\theta = (0001) \sim (\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ Dihedral ~ Poncelet problem
(Picard, R. Fuchs, Hitchin) d, g unbounded

45 exceptional solutions :

		$\frac{d}{6}$	$\frac{g}{6}$	
1	Tetrahedral			Andreev-Kitaev
7	Octahedral	6-16	0,1	(2 by Kitaev)
33	Icosahedral	5-72	0,1,2,3,7	(1 by Dubrovin 2 by Dub.-Mazzocco 2 by Kitaev)
1	'Klein'	7	0	
3	'237'	18	1	(1 by Kitaev)

Shape of table so far

[Riccatti / Rational solutions Watanabe, Mazzocco, Yuan-Li]

4 continuous families $g=0$ $d=$ 2 $\sqrt{\tau}$ (Picard/Okamoto 3)
3
4
4 } Hitchin / Dubrovin

1 discete family $\theta=(0001)\sim(\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2})$ Dihedral ~ Poncelet problem
(Picard, R. Fuchs, Hitchin) d, g unbounded

30 ~~45~~ exceptional solutions : modulo quadratic transformations

			$\frac{d}{6}$	$\frac{g}{0}$	~~Andreev-Kitaev~~
0	~~1~~	Tetrahedral			
2	~~7~~	Octahedral	6–16	0,1	1 (~~6~~ by Kitaev)
24	~~33~~	Icosahedral	5–72	0,1,2,3,7	(1 by Dubrovin 2 by Dub.–Mazzocco 2 by Kitaev)
1	~~1~~	'Klein'	7	0	
3	~~3~~	'237'	18	1	(1 by Kitaev)

Construction problem divides in two (roughly speaking):

(a) Finding solutions topologically

(b) Constructing topological solutions explicitly

Methods:

(a) 1. Finite monodromy groups $\begin{cases} SL_2 \\ GL_3 \end{cases}$

2. Pullbacks

(b) 1. Algebraic geometry [Twistors, Poncelet, Frobenius Mfds, Elliptic fibrations...]

2. Pullbacks

3. Asymptotics (à la Jimbo)

Relations $P_{VI} \leftrightarrow$ Schlesinger's equations

[Two Fuchsian Lax pairs]

Suppose $A_1(t)$, $A_2(t)$, $A_3(t)$ solve Schlesinger's eqns

$$\frac{dA_1}{dt} = \frac{[A_2, A_1]}{t}, \quad \frac{dA_3}{dt} = \frac{[A_2, A_3]}{t-1}, \quad \sum_1^3 A_i \text{ constant}$$

then ⓪ The linear connection

$$A = \left(\frac{A_1}{z} + \frac{A_2}{z-t} + \frac{A_3}{z-1} \right) dz$$

varies isomonodromically

and (if $\sum_1^3 A_i$ diagonal)

① If $A_i \in sl_2(\mathbb{C})$, the value $y(t)$ of z where

(Jimbo–Miwa) $\left(z(z-t)(z-1) A \right)_{12}$ is zero

solves P_{VI} with parameters $\underset{\sim}{\theta}$ s.t.

A_i has evals $\pm \theta_i/2$, $\sum_1^3 A_i = \begin{pmatrix} -\theta_4 & \\ & \theta_4 \end{pmatrix} / 2$

② If A_i 3×3 rank 1 then the value $y(t)$ of z where $\left(z(z-t)(z-1)A\right)_{23}$ is zero solves P_{VI} with parameters

$$\underset{\sim}{\theta} = (\lambda_1-\mu_1, \lambda_2-\mu_1, \lambda_3-\mu_1, \mu_3-\mu_2)$$

where $\lambda_i = \mathrm{Tr}(A_i)$ $i=1,2,3$

$$\Sigma_1^3 A_i = \begin{pmatrix} \mu_1 & & \\ & \mu_2 & \\ & & \mu_3 \end{pmatrix} \quad \text{so } \Sigma\lambda_i = \Sigma\mu_i$$

[Proc LMS (3) 90, 2005]

Note Positions of zeros of other 5 off diag. entries also solve P_{VI} – by ②, conjugate A by permutation matrix

– param.s change by corresponding permutation of μ's

– observe swapping μ_1 & μ_3 $\Rightarrow$

$$\underset{\sim}{\theta} \longmapsto (\theta_1-\phi, \theta_2-\phi, \theta_3-\phi, \theta_4-\phi)$$

$$\phi = \mu_3-\mu_1 = \Sigma_1^4 \theta_i/2$$

– get direct geometrical interpretation of main Okamoto trfm.

If $y(t)$ solves $\mathrm{P_{VI}}$ with parameters

$$\theta_1 = \lambda_1 - \mu_1, \quad \theta_2 = \lambda_2 - \mu_1, \quad \theta_3 = \lambda_3 - \mu_1, \quad \theta_4 = \mu_3 - \mu_2$$

and we define $x(t)$ via

$$x = \frac{1}{2}\left(\frac{t(t-1)y'}{y(y-1)(y-t)} - \frac{\theta_1}{y} - \frac{\theta_3}{y-1} - \frac{\theta_2+1}{y-t}\right)$$

then the family of Fuchsian systems

$$\frac{d}{dz} - \left(\frac{B_1}{z} + \frac{B_2}{z-t} + \frac{B_3}{z-1}\right)$$

will be isomonodromic as t varies, where

$$B_1 = \begin{pmatrix} \lambda_1 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \lambda_2 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \lambda_3 \end{pmatrix}$$

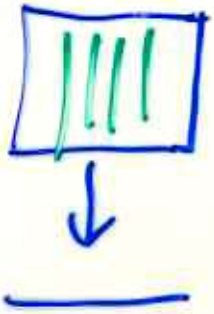
$$b_{12} = \lambda_1 - \mu_3 y + (\mu_1 - xy)(y-1), \qquad b_{32} = (\mu_2 - \lambda_2 - b_{12})/t,$$

$$b_{13} = \lambda_1 t - \mu_3 y + (\mu_1 - xy)(y-t), \qquad b_{23} = (\mu_2 - \lambda_3)t - b_{13},$$

$$b_{21} = \lambda_2 + \frac{\mu_3(y-t) - \mu_1(y-1) + x(y-t)(y-1)}{t-1}, \qquad b_{31} = (\mu_2 - \lambda_1 - b_{21})/t.$$

[6 results on P_6, math.AG/0503043]

Aside on flat connections



$$\begin{array}{c} M \\ \pi \downarrow \\ B \end{array}$$

Fibre bundle, std fibre F

locally a product

$M_b = \pi^{-1}(b) \cong F \quad (\forall\, b \in B)$

small $U \subset B \Rightarrow \pi^{-1}(U) \cong U \times F$

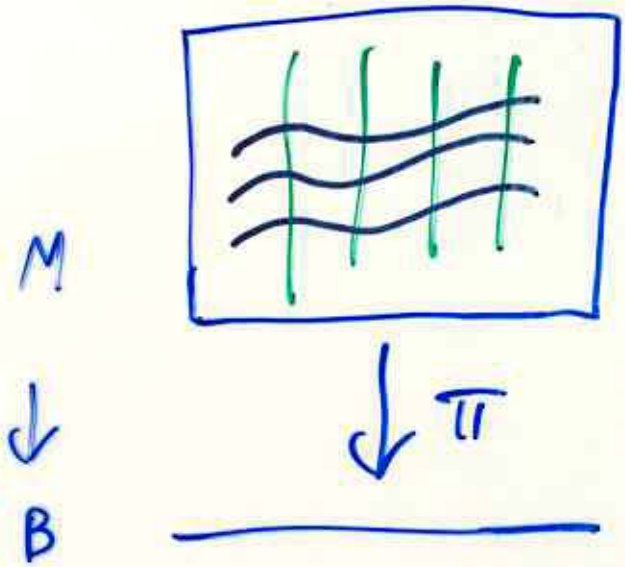
local trivialization

- Complete flat connection on $M \to B$ is a way to identify 'nearby' fibres canonically:

  If $U \subset B$ open ball (i.e contractible) get isomorphisms $M_{b_1} \cong M_{b_2} \quad \forall\, b_1, b_2 \in U$

  – so get natural choice of local trivialization / U

- The connection is the infinitesimal object giving these isomorphisms



Horizontal lines (sections of π)
~ identifications of fibres
– determined by their tangent spaces
$H_p \subset T_pM$ $(p \in M)$

If $p \in M$ have vertical subspace

$$V_p = T_p M_{\pi(p)}$$

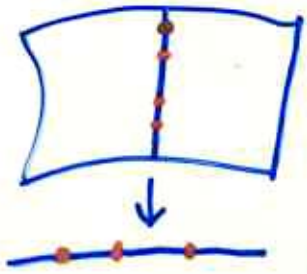
– connection is choice of field of horizontal subspaces H_p transverse to V_p

$$T_p = V_p \oplus H_p$$

– choose coords on fibres & base $\Rightarrow$
1st order system of coupled (nonlinear) differential equations

Conceptual approach

Consider universal family of $\mathbb{P}^1$'s with 4 punctures (ordered)



$$\begin{tikzcd} C \arrow[d] & \mathcal{F} \cong \mathbb{P}^1 \setminus 4 \text{ points} \arrow[l, hook'] \\ B := \mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0,1,\infty\} & \end{tikzcd}$$

- Replace each fibre $\mathcal{F}$ by $H^1(\mathcal{F}, G)$, $G = SL_2(\mathbb{C})$

Two viewpoints here on H^1 :-

Betti Moduli of π_1 representations $\mathrm{Hom}(\pi_1(\mathcal{F}), G)/G$

$\wr\uparrow$ Riemann-Hilbert

DeRham Moduli of connections on holomorphic vector bundles

- Get two (nonlinear) fibrations over $B = \mathcal{M}_{0,4}$
- As in abelian case get flat connection on bundle (now nonlinear connection)

$$\begin{tikzcd} \mathcal{M}_{DR} \arrow[r, "RH"] \arrow[d] & \mathcal{M}_{Betti} \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Two descriptions of connection:

- Betti (periods $\rightsquigarrow$ monodromy)

  - keep monodromy representation constant

- DeRham (one forms $\rightsquigarrow$ connections on vector bundles
  closed $\rightsquigarrow$ flat)

  - extend flat connection on fibre $\mathcal{A}$ to full flat connection on family of fibres & restrict to another fibre

Applications DR $\rightsquigarrow$ explicit nonlinear equations $\rightsquigarrow$ P_{VI}

Betti $\rightsquigarrow$ explicit description of monodromy of nonlinear connection

Explicit nonlinear equations

$\mathcal{M}_{DR}$ well approximated by moduli of log. connections on trivial bundles over $\mathbb{P}^1$:-

$$\mathcal{M}^* \cong \left\{ d - \sum_1^3 \frac{A_i}{z-a_i} dz \right\} / \text{isomorphism}$$

$$\cong \{ (A_1, \ldots, A_4) \mid A_i \in \mathfrak{g},\ \textstyle\sum_1^4 A_i = 0 \} / G \quad \times B$$

Nonlinear connection on $\begin{matrix} \mathcal{M}^* \\ \downarrow \\ B \end{matrix}$ was computed by Schlesinger:

Horizontal sections satisfy $\quad \dfrac{\partial A_i}{\partial a_j} = \dfrac{[A_i, A_j]}{a_i - a_j} \quad i \neq j$

~ flatness of full connection $\quad d - \sum A_i \dfrac{d(z-a_i)}{z-a_i}$

Note
- fibres of $\mathcal{M}^*/B$ 6d Poisson manifolds
- Schlesinger's equations preserve the eigenvalues of each A_i $(i=1,2,3,4)$
- flows restrict to 2d symplectic leaves
- choose coords x, y on leaves $\Rightarrow$ coupled 1st order ODEs
- eliminate $x \Rightarrow$ 2nd order ode for $y(t)$ – Painlevé VI (t = coord on $B = \mathcal{M}_{0,4}$)

Monodromy of Painlevé VI

$\sim$ monodromy of connection on $\begin{array}{c} M_{Betti} \\ \downarrow \\ B \end{array}$

- connection is complete & flat so $\Leftrightarrow$

action of $\pi_1(B) \cong F_2 \circlearrowleft$ fibre M_t

$$\begin{array}{c} \text{fibre } M_t \\ \| \\ \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus 4\text{points}), G)/G \end{array}$$

Given choice of loops generating $\pi_1(\mathbb{P}^1 \setminus 4\text{points})$ get

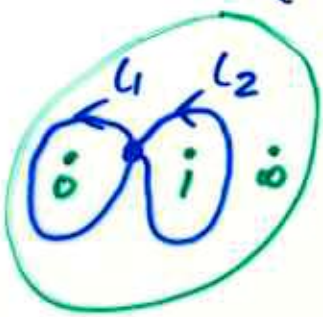
$$M_t \cong \{(M_1 M_2 M_3 M_4) \mid M_i \in G,\ M_4 M_3 M_2 M_1 = 1\}/G$$
$$\cong G^3/G$$

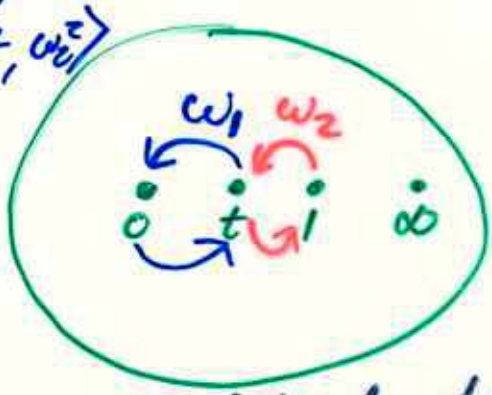
– universal family of cubic surfaces (Fricke-Klein/Cayley)

What is the monodromy action $\pi_1(B) \circlearrowright M_t$?

$$\pi_1(B) \cong \mathcal{F}_2 \cong \text{Pure mapping class group of } (\mathbb{P}^1, 0, t, 1, \infty)$$

$$\langle l_1, l_2 \rangle \xrightarrow[l_i \mapsto \omega_i^2]{\sim} \langle \omega_1^2, \omega_2^2 \rangle$$





Dehn twists

- Mapping class gp acts naturally on $M_t = \mathrm{Hom}_{\underline{e}}(\pi_1(\mathbb{P}^1 \setminus \{0,t,1,\infty\}), G)/G$

  by "pushing forward loops"

$$\left[f(\rho)(\gamma) = \rho(f \circ \gamma) \qquad \begin{array}{l} \rho \in M_t \\ \gamma \in \pi_1 \\ f : \mathbb{P}^1 \to \mathbb{P}^1 \text{ diffeo} \end{array} \right]$$

- This is the monodromy action of $\boxed{B_{VI}}$
- Explicitly on monod. matrices:

$$\omega_1(M_1, M_2, M_3) = (M_2, M_2 M_1 M_2^{-1}, M_3)$$

$$\omega_2(\text{———}) = (M_1, M_3, M_3 M_2 M_3^{-1})$$

Definition

Topological algebraic P_{VI} solution is an $\mathcal{T}_2$ orbit of triples (M_1, M_2, M_3) of monodromy matrices which is <u>finite</u>

- clearly algebraic solutions have finite monodromy (& so finite $\mathcal{T}_2$ orbits)
- In 2×2 case $M_i \in SL_2(\mathbb{C})$
- In 3×3 case M_i should be a pseudo-reflection (of form "1 + rank 1")

Obvious topological solutions :-

(*) [If M_1, M_2, M_3 generate a finite group then $\mathcal{T}_2$ orbit must be finite] (*)

Topological solution $\Rightarrow$ map $t : \Pi \to \mathbb{P}^1$
(topologically)

Idea: deg d rational maps $t : \Pi \to \mathbb{P}^1$
with r branch points

$\Big\downarrow \cong$ (remove branchpoints)

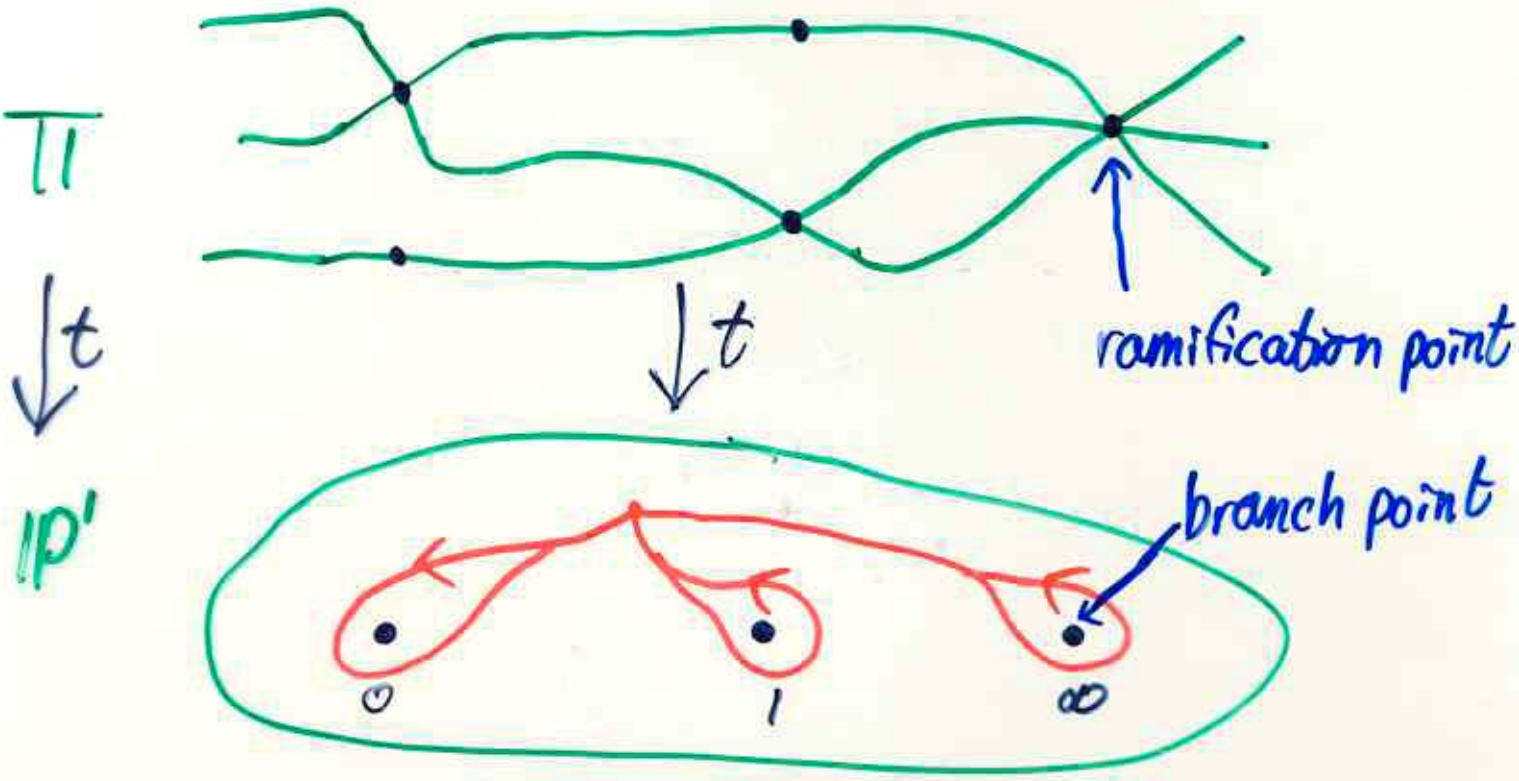
bundles over $\mathbb{P}^1 \setminus (r \text{ points})$
with fibres F = finite set with d-points

$\Big\downarrow \cong$ (take monodromy)

representations

$$\begin{array}{ccc} \pi_1(\mathbb{P}^1 \setminus r \text{ points}) & \longrightarrow & \mathrm{Aut}(F) \\ \wr\| & & \wr\| \\ \hat{F}_{r-1} & \longrightarrow & \mathrm{Sym}_d \end{array}$$

Here $r=3$ (Belyi map) so cover determined
by $\sigma_0, \sigma_1, \sigma_\infty \in \mathrm{Sym}_d$
$(\sigma_\infty \sigma_1 \sigma_0 = 1)$



For us: fibre of t is $F = \{ \mathfrak{A}_2 \text{ orbit of } (M_1, M_2, M_3) \}$ / conjugation

& $\mathfrak{A}_2$ action on F gives σ_0, σ_1

Riemann–Hurwitz $\Rightarrow$ genus of Π:

$$2g(\Pi) - 2 = d(2g(\mathbb{P}^1) - 2) + \sum_{\substack{\text{ramification} \\ \text{points}}} (i_r - 1)$$

i.e. $$g(\Pi) = 1 - d + \tfrac{1}{2}\sum (i_r - 1)$$

$= 1 - 3 + 2 = 0$ in example

[Ramification indices = cycle lengths of permutations $\sigma_0, \sigma_1, \sigma_\infty$

Can now go through list of triples of generators of

- finite subgroups $SL_2(\mathbb{C})$
- 3d complex reflection groups (generators should be reflections)

- Started by Hitchin

$SL_2(\mathbb{C})$, triples of form (g, g^{-1}, hgh^{-1})

generating binary dihedral, tetra-, octahedral gps

- All interesting dihedral solutions [Poncelet problem]
- 3/4 branch tet/oct solutions

- Effectively Dubrovin-Mazzocco did case of 3d real reflection group (orthogonal)

- 3/4 branch tet/oct solns (equiv. to Hitchins)
- 3 icosahedral solns $d = 10, 10, 18$, $g = 0, 0, 1$

(equiv. to solutions w. finite $SL_2(\mathbb{C})$ monodromy) 10 pages!

$\theta = (0,0,0,k) \sim (k,k,k,k)/2$

Dihedral solutions

- interesting solutions have $\theta = (1/2, 1/2, 1/2, 1/2)$
- equivalent to P_{VI} equation completely solved by Picard / R. Fuchs
- transcendental formula

<u>Qn:</u> explicit algebraic formula for π, y, t?

- solved by Hitchin '96 (determinental formula) using Cayley's solution of Poncelet problem, Barth-Michel's parameterisation of modular curves...

- e.g D_5 – first explicit elliptic solution

[later M. Mazzocco looked at dihedral reflection groups – solutions equiv to those above.

Elliptic dihedral solution

Hitchin 1996

12 branches, genus 1

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/2, 1/2)$$

$$y = \frac{(3\,s-1)\left(s^2-4\,s-1\right)\left(s^2+u\right)\left(s\left(s+2\right)-u\right)}{\left(3\,s^3+7\,s^2+s+1\right)\left(s^2-u\right)\left(s\left(s-2\right)+u\right)}$$

$$t = \frac{\left(s^2+u\right)^2\left(s\left(s+2\right)-u\right)\left(s\left(s-2\right)-u\right)}{\left(s^2-u\right)^2\left(s\left(s+2\right)+u\right)\left(s\left(s-2\right)+u\right)}$$

where s, u satisfy:

$$u^2 = s\left(s^2+s-1\right)$$

(Triply generated) 3d complex reflection groups

Dihedral

Tetrahedral

Octahedral

Icosahedral

$G(m, p, 3)$

Klein $\quad 2\,PSL_2(7)$

Hesse 1

Hesse 2

Valentiner $\quad 6\,A_6$

[Shephard–Todd 1954]

Topological Klein solution

- upto equivalence Klein reflection group has 1 triple of generating reflections:

$$r_1 = \frac{1}{2}\begin{pmatrix} 1 & -1 & a \\ -1 & 1 & a \\ \bar{a} & \bar{a} & 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

where $a = \frac{1}{2}(-1 + i\sqrt{7})$

- $\mathbb{Z}_2$ orbit (on conjugacy classes of triples) has size 7 — so degree $d = 7$

- permutation $\sigma_0, \sigma_1, \sigma_\infty$ each have cycle type $(2,2,3)$

  - so genus $(\Pi) = 1 - 7 + 3 \cdot 4/2 = 0$

- construction?

Similarly Valentiner group $\rightsquigarrow$ 3 genus 1 solutions degrees $= 15, 15, 24$