Existing construction methods

① "Algebraic geometry"

- Hitchin: find (solved) classical problem
- Dubrovin: Frobenius manifolds

② Pullbacks – Kitaev / Doran

- need ansatz & hope computer can find parameterized solution of N algebraic equations in $N+1$ unknowns (Kitaev has some examples)

③ Exact asymptotics

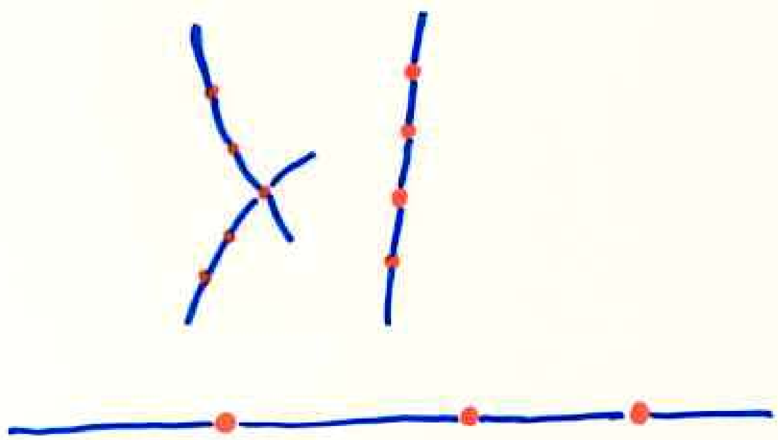
- Dubrovin-Mazzocco on line $\theta = (0,0,0,k)$ (proved asymptotic formula for such θ's) + classified all such solutions (5 + dihedral)

Key inputs

1 Jimbo

- leading asymptotics of P_{VI} solution y in terms of linear monodromy (M_1, M_2, M_3)

Degenerate to stable curve & solve Riemann-Hilbert problems there:



$\overline{B} = \overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$

Theorem. (Jimbo 1982)

Suppose we have four matrices $M_j \in \mathrm{SL}_2(\mathbb{C})$, $j = 1, 2, 3, 4$ satisfying

a) $M_4 M_3 M_2 M_1 = 1$,

b) M_j has eigenvalues $\{\exp(\pm \pi i \theta_j)\}$ with $\theta_j \notin \mathbb{Z}$,

c) $\mathrm{Tr}(M_1 M_2) = 2\cos(\pi\sigma)$ for some nonzero $\sigma \in \mathbb{C}$ with $0 \leq \mathrm{Re}(\sigma) < 1$,

d) None of the eight numbers

$$\theta_1 \pm \theta_2 \pm \sigma, \quad \theta_1 \pm \theta_2 \mp \sigma, \quad \theta_4 \pm \theta_3 \pm \sigma, \quad \theta_4 \pm \theta_3 \mp \sigma$$

is an even integer.

Then the leading term in the asymptotic expansion at zero of the corresponding Painlevé VI solution $y(t)$ on the branch corresponding to $[(M_1, M_2, M_3)]$ is

$$\left(\frac{(\theta_1 + \theta_2 + \sigma)(-\theta_1 + \theta_2 + \sigma)(\theta_4 + \theta_3 + \sigma)}{4\sigma^2(\theta_4 + \theta_3 - \sigma)\widehat{s}} \right) t^{1-\sigma}$$

where

$$\widehat{s} = c \times s, \qquad s = \frac{a + b}{d}$$

$$a = e^{\pi i \sigma}(i \sin(\pi\sigma)\cos(\pi\sigma_{23}) - \cos(\pi\theta_2)\cos(\pi\theta_4) - \cos(\pi\theta_1)\cos(\pi\theta_3))$$

$$b = i \sin(\pi\sigma)\cos(\pi\sigma_{13}) + \cos(\pi\theta_2)\cos(\pi\theta_3) + \cos(\pi\theta_4)\cos(\pi\theta_1)$$

$$d = 4\sin\left(\frac{\pi}{2}(\theta_1 + \theta_2 - \sigma)\right)\sin\left(\frac{\pi}{2}(\theta_1 - \theta_2 + \sigma)\right)\sin\left(\frac{\pi}{2}(\theta_4 + \theta_3 - \sigma)\right)\sin\left(\frac{\pi}{2}(\theta_4 - \theta_3 + \sigma)\right)$$

$$c = \frac{(\Gamma(1-\sigma))^2 \widehat{\Gamma}(\theta_1 + \theta_2 + \sigma)\widehat{\Gamma}(-\theta_1 + \theta_2 + \sigma)\widehat{\Gamma}(\theta_4 + \theta_3 + \sigma)\widehat{\Gamma}(-\theta_4 + \theta_3 + \sigma)}{(\Gamma(1+\sigma))^2 \widehat{\Gamma}(\theta_1 + \theta_2 - \sigma)\widehat{\Gamma}(-\theta_1 + \theta_2 - \sigma)\widehat{\Gamma}(\theta_4 + \theta_3 - \sigma)\widehat{\Gamma}(-\theta_4 + \theta_3 - \sigma)}$$

where $\widehat{\Gamma}(x) := \Gamma(\frac{1}{2}x + 1)$ (with Γ being the usual gamma function) and where $\sigma_{jk} \in \mathbb{C}$ $(j, k \in \{1, 2, 3\})$ is determined by $\mathrm{Tr}(M_j M_k) = 2\cos(\pi\sigma_{jk})$, $0 \leq \mathrm{Re}(\sigma_{jk}) \leq 1$, so $\sigma = \sigma_{12}$.

2 Relate systems A & B on both DeRham & Betti sides

- monodromy changes in highly non-trivial way:

$$\text{Klein reflection group} \rightsquigarrow \Delta_{237}$$

$$\text{Valentiner group} \rightsquigarrow A_5$$

- apparently procedure is complex analytic version of N. Katz's "middle convolution functor"

Sketch [Balser et al 1981, — Proc LMS 2005]

Fix distinct $a_1, \dots, a_n \in \mathbb{C}$

Let $A_0 = \operatorname{diag}(a_1, \dots, a_n)$

$$
\begin{tikzcd}
 & A \in \mathfrak{gl}_n(\mathbb{C}) \arrow[dl, "A_i = E_i A"'] \arrow[dr] & \\
d - \sum_1^n \frac{A_i}{z-a_i} dz \arrow[rr, leftrightarrow, "\text{Fourier-Laplace}"'] \arrow[d, "\text{Riemann-Hilbert}"] & & d - \left(\frac{A_0}{z^2} + \frac{A}{z}\right) dz \arrow[d, "\text{Stokes}"'] \\
r_1, \dots, r_n,\ r_i = 1 + e_i \otimes \alpha_i \arrow[rr, leftrightarrow, "\text{Killing-Coxeter}", "(\alpha_i(e_j)) = t u_+ - u_-"'] \arrow[dr, "r_n \cdots r_1"'] & & u_+, u_-, t \in U_+ \times U_- \times T \arrow[dl, "u_-^{-1} t u_+"] \\
 & G &
\end{tikzcd}
$$

Scalar shift $A \longmapsto A + \lambda I$

- tensor by $\lambda \frac{dz}{z}$ on RHS
- nontrivial convolution on LHS

$n=3$: choose λ s.t. $A+\lambda$ rank $2 \Rightarrow$ reducible on LHS

- take 2×2 quotient connection $\rightsquigarrow SL_2$ connection

$\Rightarrow$ $\mathbb{Z}_2$ equivariant maps: 3×3 triples $\mapsto$ 2×2 triples

3d reflection group	Subgroup $Sl_2(\mathbb{C})$
Tetrahedral	Octahedral
Octahedral	Tetrahedral
Icosahedral ($d=10,10,18$)	Icosahedral
Klein	Δ_{237}
Valentiner ($d=15,15,24$)	Icosahedral (!)

Topological solution

$\Downarrow$ Jimbo

Leading asymptotics at $t=0$ on each branch

$\Downarrow$ substitute back into P_{VI}

Any no. of terms of Puiseux expansion at 0 of y on each branch

$\Downarrow$ Finite no. coeffs to determine

Solution polynomial $F(y,t)$

$\Downarrow$ Maple / help from M. van Hoeij

Parameterised solution Π, y, t

[useful tricks listed in math.DG/0501464 $\rightsquigarrow$ Bolibruch's volume]

Klein solution

seven branches

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (2/7, 2/7, 2/7, 4/7)$$

$$y = -\frac{(5\,s^2 - 8\,s + 5)\,(7\,s^2 - 7\,s + 4)}{s\,(s-2)\,(s+1)\,(2\,s-1)\,(4\,s^2 - 7\,s + 7)}$$

$$t = \frac{(7\,s^2 - 7\,s + 4)^2}{s^3\,(4\,s^2 - 7\,s + 7)^2}.$$

Corollary

For any s such that $t(s) \neq 0, 1, \infty$ the family of Fuchsian systems

$$\frac{d}{dz} - \left(\frac{B_1}{z} + \frac{B_2}{z - t(s)} + \frac{B_3}{z-1}\right)$$

has monodromy isomorphic to the Klein complex reflection group, where

$$B_1 = \begin{pmatrix} \frac{1}{2} & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \frac{1}{2} & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \frac{1}{2} \end{pmatrix}$$

$$b_{12} = \frac{14\,s^3 - 21\,s^2 + 24\,s - 22}{21s\,(4\,s^2 - 7\,s + 7)}, \qquad b_{13} = \frac{22\,s^3 - 24\,s^2 + 21\,s - 14}{21(7\,s^2 - 7\,s + 4)},$$

$$b_{21} = \frac{14\,s^3 - 21\,s^2 + 24\,s + 5}{21\,(s-1)\,(4\,s^2 - s + 4)}, \qquad b_{23} = \frac{22\,s^3 - 42\,s^2 + 39\,s - 5}{21(7\,s^2 - 7\,s + 4)},$$

$$b_{31} = \frac{14 - 21\,s + 24\,s^2 + 5\,s^3}{21\,(s-1)\,(4\,s^2 - s + 4)}, \qquad b_{32} = \frac{22 - 42\,s + 39\,s^2 - 5\,s^3}{21s\,(4\,s^2 - 7\,s + 7)}.$$

Icosahedral Classification

Γ = binary icosahedral group $\subset SL_2(\mathbb{C})$

Prop. (P. Hall)

Up to conjugacy Γ has 26,688 triples of generators

Problem: Classify topological icosahedral solutions upto Okamoto's $W_a(F_4)$ action

Trick: bound above & below

Let $S = \{(M_1, M_2, M_3) \mid M_i \in \Gamma, \langle M_i \rangle = \Gamma\} / \Gamma$

(so $\# S = 26688$)

Have map to θ-parameters

$$S \xrightarrow{p} \mathbb{Q}^4 \subset \mathbb{R}^4$$

$$(M_1, M_2, M_3) \longmapsto (\theta_1, \theta_2, \theta_3, \theta_4)$$

s.t M_j has eigenvalues $\exp(\pm \pi i \theta_j)$
& $\theta_j \in [0,1]$, $M_4 = (M_3 M_2 M_1)^{-1}$

Definition $T_1, T_2 \in S$ are parameter equivalent if $p(T_1)$ & $p(T_2)$ are in the same orbit of the standard $W_a(F_4)$ action (on $\mathbb{R}^4$)

Proposition
- Okamoto equivalence $\Rightarrow$ parameter equivalence
- S maps to exactly 52 parameter equivalence classes

Geometric Equivalence

Recall $\mathcal{F}_2 = \pi_1(B) \cong$ pure mapping class group of $\mathbb{P}^1$ w. 4 marked pts

– acts on S

Let MC = full mapping class group

$$1 \longrightarrow \mathcal{F}_2 \longrightarrow MC \longrightarrow Sym_4 \longrightarrow 1$$

– MC also acts on S

Also let $\Sigma = (\mathbb{Z}/2)^3 = \{(\pm 1, \pm 1, \pm 1)\}$

–acts on S in obvious way ($n_i \mapsto \pm n_i$)

$\Rightarrow$ group $\widetilde{MC} := MC \ltimes \Sigma$ acts on S

Def^{n}: orbits of $\widetilde{MC}$ = geometric equivalence classes

Prop.
- Geom. equiv. $\Rightarrow$ Okamoto equiv (delicate)
- $\widetilde{MC}$ has exactly 52 orbits on S

Corollary $\exists$ exactly 52 inequivalent icosahedral solutions to P_{VI} [–, Crelle 596 '06]

Icosahedral solutions with $\leqslant 4$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)	
1	1	0	1	$a\,b\,c$	31, 19, 11, 1	192	1	
2	1	0	1	$a\,b\,d$	37, 17, 13, 7	192	1	
3	1	0	1	$a\,c\,d$	33, 21, 9, 3	192	1	
4	1	0	1	$b\,c\,d$	28, 16, 8, 4	192	1	
5	1	0	2	$b^2\,c$	26, 14, 6, 6	96	1	Schwarz
6	1	0	2	$b^2\,d$	38, 18, 18, 2	96	1	$(y=t)$
7	1	0	2	$b\,c^2$	22, 10, 10, 2	96	1	
8	1	0	2	$b\,d^2$	34, 14, 10, 10	96	1	
9	1	0	3	c^3	18, 6, 6, 6	32	1	
10	1	0	3	d^3	42, 18, 18, 6	32	1	
11	2	0	2	$b^2\,c^2$	42, 18, 10, 10	96	2	
12	2	0	2	$b^2\,d^2$	50, 10, 6, 6	96	2	$\sqrt{t}$
13	2	0	2	$c^2\,d^2$	42, 18, 6, 6	96	2	
14	3	0	1	$b\,c^2\,d$	40, 16, 8, 8	288	S_3	Tet. family
15	3	0	1	$b\,c\,d^2$	40, 8, 4, 4	288	S_3	
16	4	0	2	$a\,c^3$	33, 9, 9, 9	128	A_4	Dih. family
17	4	0	2	$a\,d^3$	51, 3, 3, 3	128	A_4	
18	4	0	2	$c^3\,d$	30, 6, 6, 6	128	A_4	Oct. family
19	4	0	2	$c\,d^3$	42, 6, 6, 6	128	A_4	

Icosahedral solutions with $\geqslant 5$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)	
20	5	0	1	$b^2\,c\,d$	44, 12, 12, 4	480	S_5	
21	5	0	2	$c^2\,d^2$	36, 12, 0, 0	240	S_5	Kitaev
22	6	0	1	$b\,c^2\,d$	34, 10, 2, 2	576	S_6	
23	6	0	1	$b\,c\,d^2$	46, 14, 10, 2	576	S_6	
24	8	0	1	$a\,c^2\,d$	39, 15, 3, 3	768	A_8	
25	8	0	1	$a\,c\,d^2$	45, 9, 9, 3	768	A_8	
26	9	1	2	$b\,c^3$	28, 4, 4, 4	288	A_9	Kitaev
27	9	1	2	$b\,d^3$	52, 8, 8, 4	288	A_9	
28	10	0	2	$a^2\,c\,d$	48, 12, 6, 6	480	$2^7\,3\,5$	
29	10	0	2	$b^3\,c$	46, 14, 14, 6	320	A_{10}	
30	10	0	2	$b^3\,d$	42, 2, 2, 2	320	A_{10}	
31	10	0	3	c^4	24, 0, 0, 0	80	A_{10}	DM
32	10	0	3	d^4	48, 0, 0, 0	80	A_{10}	
33	12	0	0	$a\,b\,c\,d$	43, 11, 7, 1	2304	A_{12}	
34	12	1	1	$a\,b\,c^2$	37, 13, 5, 5	1152	A_{12}	
35	12	1	1	$a\,b\,d^2$	49, 5, 5, 1	1152	A_{12}	
36	12	1	1	$b^2\,c\,d$	38, 6, 6, 2	1152	$2^9\,3^2\,5$	
37	15	1	2	$b^3\,c$	36, 4, 4, 4	480	A_{15}	Valentiner
38	15	1	2	$b^3\,d$	48, 8, 8, 8	480	A_{15}	
39	15	1	2	$b^2\,c^2$	32, 8, 0, 0	720	S_{15}	
40	15	1	2	$b^2\,d^2$	44, 4, 0, 0	720	S_{15}	
41	18	1	3	b^4	40, 0, 0, 0	144	$2^{14}\,3^4\,5\,7$	DM
42	20	1	1	$a\,b^2\,c$	41, 9, 9, 1	1920	A_{20}	
43	20	1	1	$a\,b^2\,d$	47, 7, 3, 3	1920	A_{20}	
44	20	1	3	$a^2\,c^2$	42, 18, 0, 0	480	$2^{17}\,3^4\,5^2\,7$	
45	20	1	3	$a^2\,d^2$	54, 6, 0, 0	480	$2^{17}\,3^4\,5^2\,7$	
46	24	1	2	$a\,b^3$	45, 5, 5, 5	768	$2^{20}\,3^5\,5^2\,7\,11$	Valentiner
47	30	2	2	$a^2\,b\,c$	46, 14, 4, 4	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$	
48	30	2	2	$a^2\,b\,d$	52, 8, 2, 2	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$	
49	36	3	3	$a^2\,b^2$	50, 10, 0, 0	864	$2^{23}\,3^4\,5\,7$	
50	40	3	3	$a^3\,c$	51, 9, 9, 9	320	$2^{25}\,3^4\,5^2\,7$	
51	40	3	3	$a^3\,d$	57, 3, 3, 3	320	$2^{25}\,3^4\,5^2\,7$	
52	72	7	3	$a^3\,b$	55, 5, 5, 5	576	$2^{32}\,3^4\,5\,7$	

Solution 20, genus zero, 5 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 1/3, 1/5, 2/3)$:

$$y = \frac{2\,(s^2+s+7)\,(5\,s-2)}{s\,(s+5)\,(4\,s^2-5\,s+10)}, \qquad t = \frac{27\,(5\,s-2)^2}{(s+5)\,(4\,s^2-5\,s+10)^2}$$

Solution 24, genus zero, 8 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 2/5, 1/5, 4/5)$:

$$y = \frac{s\,(s+4)\,(3\,s^4-2\,s^3-2\,s^2+8\,s+8)}{8\,(s-1)\,(s^2+4)\,(s+1)^2}, \qquad t = \frac{s^5\,(s+4)^3}{4\,(s-1)\,(s^2+4)^2\,(s+1)^3}$$

Solution 25, genus zero, 8 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 2/5, 1/2, 4/5)$:

$$y = \frac{s^2\,(5\,s^3+2\,s^2-4\,s-8)\,(s+4)^2}{4\,(s+1)^2\,(s^2+4)\,(s-1)\,(s^2+3\,s+6)}, \qquad t = \frac{s^5\,(s+4)^3}{4\,(s-1)\,(s^2+4)^2\,(s+1)^3}$$

Solution 28, genus zero, 10 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/5, 3/5)$:

$$y = \frac{(s^5+5\,s^4-20\,s^3+75\,s+75)\,(s^2-5)\,(s^2+5)}{(s+1)^2\,(s^2-4\,s+5)\,(s+5)\,(s^4+6\,s^2-75)}, \quad t = \frac{2\,(s^2+5)^3\,(s^2-5)^2}{(s+5)^3\,(s^2-4\,s+5)^2\,(s+1)^3}$$

"Generic" solution, genus zero, 12 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (2/5, 1/2, 1/3, 4/5)$:

$$y = -\frac{9s(s^2+1)(3s-4)(15s^4-5s^3+3s^2-3s+2)}{(2s-1)^2(9s^2+4)(9s^2+3s+10)}$$

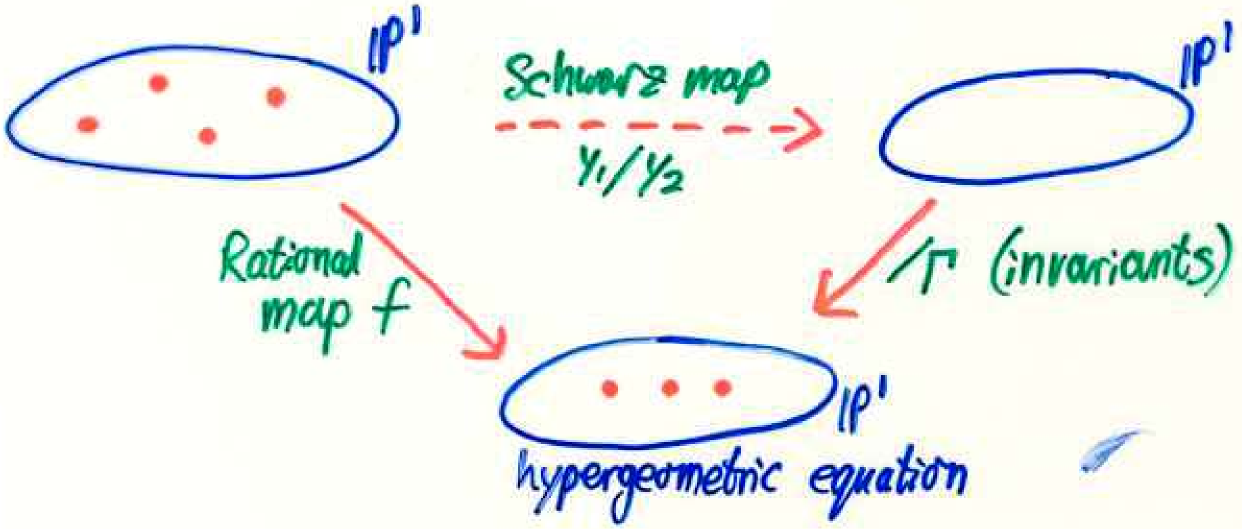
$$t = \frac{27s^5(s^2+1)^2(3s-4)^3}{4(2s-1)^3(9s^2+4)^2}$$

$$F(y,t) =$$

$$(15524784\,t^2 - 5373216\,t + 1350000)\,y^{12} - (128381760\,t^2 - 13366080\,t)\,y^{11} +$$
$$(5425704\,t^3 + 496677744\,t^2 - 30539160\,t)\,y^{10} -$$
$$(14929920\,t^4 + 41364000\,t^3 + 866759680\,t^2 - 2928160\,t)\,y^9 +$$
$$(107546535\,t^4 - 508275750\,t^3 + 747613335\,t^2 - 1837080\,t)\,y^8 -$$
$$(24385536\,t^5 - 285548724\,t^4 - 2437066824\,t^3 + 74927724\,t^2 + 944784\,t)\,y^7 +$$
$$(58212000\,t^5 - 2865570750\,t^4 - 4456260900\,t^3 + 17631810\,t^2)\,y^6 -$$
$$(49787136\,t^6 - 904003584\,t^5 - 7215732804\,t^4 - 2130570936\,t^3 - 12872196\,t^2)\,y^5 -$$
$$(413500320\,t^6 + 3724484160\,t^5 + 4839581265\,t^4 + 162430110\,t^3 + 3750705\,t^2)\,y^4 +$$
$$(3001304640\,t^6 + 74794560\,t^5 + 2710584000\,t^4 - 380946240\,t^3)\,y^3 -$$
$$(940800000\,t^7 + 977540640\,t^6 - 726801696\,t^5 + 939255264\,t^4 - 72013536\,t^3)\,y^2 +$$
$$(1176000000\,t^7 - 1481095680\,t^6 + 765158400\,t^5)\,y -$$
$$(1920800000\,t^8 - 7212800000\,t^7 + 10522980864\,t^6 - 6913299456\,t^5 + 1728324864\,t^4)$$

Pullbacks (Klein, R-Fuchs, ..., Kitaev, C. Doran, ...)

Klein showed all 2nd order Fuchsian equations with finite monodromy are (essentially) pullbacks of hypergeometric equations:



so isomonodromic family of ODEs ~ family of rational maps

Key observation: algebraicity of deformation comes from that of rational maps (Hurwitz spaces) not from finiteness of monodromy representation
(Doran, Kitaev)

C. Doran JDG 2001

regular singular point at λ, and precisely four non-apparent regular singular points at $\{0, 1, \infty, t\}$. The local monodromies about these points do not vary with $t \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. By Lemma 2.9, we thus know that λ as a function of t determines a solution to a Painlevé VI equation as described. q.e.d.

A direct application of this criterion to the natural hypergeometric local systems associated to triangles yields the following three corollaries:

Ramification indices over $0, 1, \infty$

Corollary 4.6. *The following is the complete list of topological types corresponding to algebraic Painlevé VI solutions coming from pullback from arithmetic Fuchsian triangle groups, together with the description of the corresponding triangle:*

Degree of rational map f

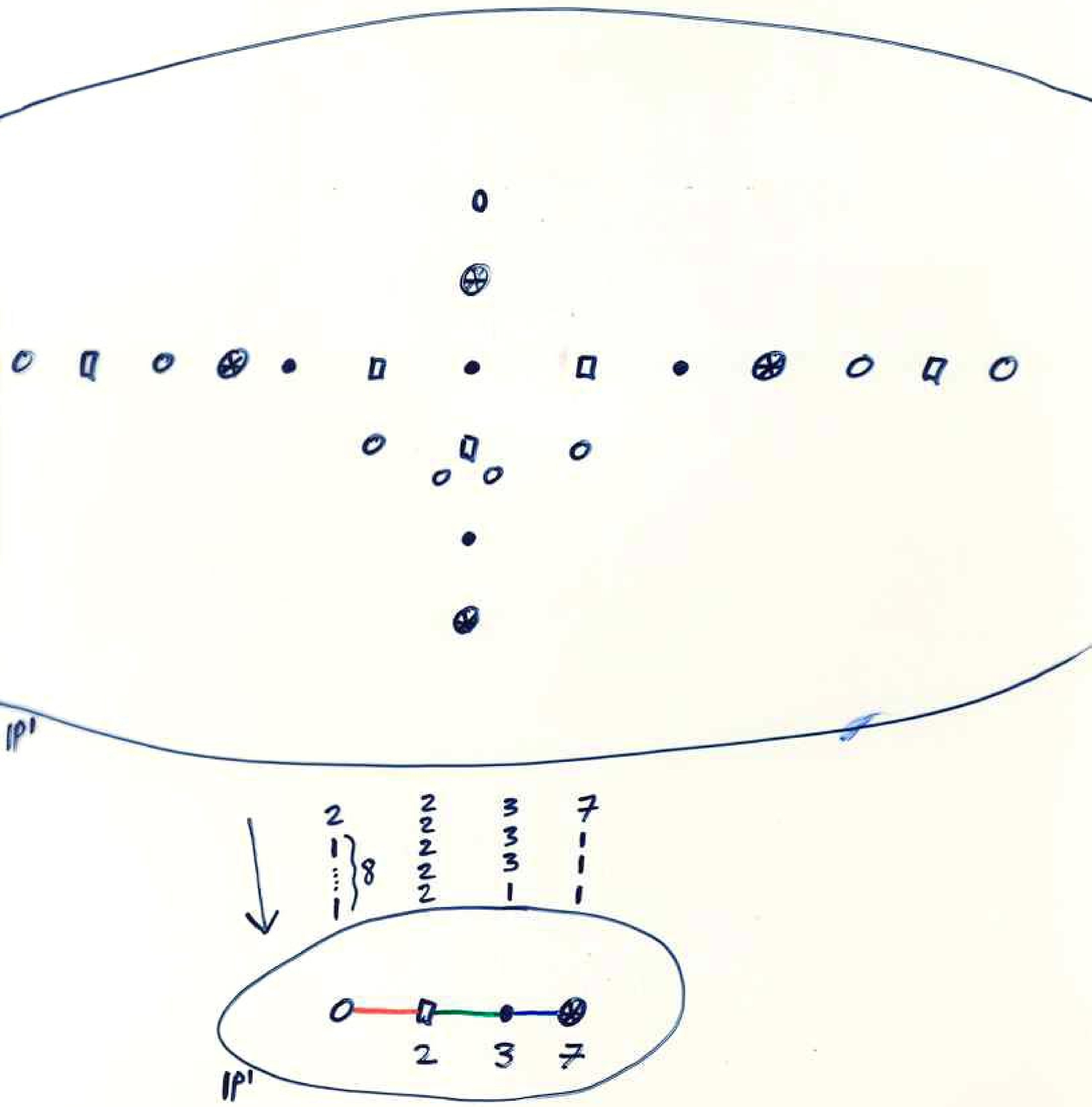
Triangle group

$(2; [2], [1, 1], [1, 1]; 2)$	$(2, \square, \square)$	
$(3; [2, 1], [3], [1, 1, 1]; 2)$	$(2, 3, \square)$	
$(4; [2, 2], [3, 1], [2, 1, 1]; 2)$	$(2, 3, \square)$	≤ 4 branches
$(4; [2, 2], [4], [1, 1, 1, 1]; 2)$	$(2, 4, \square)$	
$(6; [2, 2, 2], [3, 3], [2, 2, 1, 1]; 2)$	$(2, 3, \square)$	
$(6; [2, 2, 2], [3, 3], [3, 1, 1, 1]; 2)$	$(2, 3, \square)$	
$(10; [2, \ldots, 2], [3, 3, 3, 1], [7, 1, 1, 1]; 2)$	$(2, 3, 7)$	$g=1$, $d=18$ new
$(12; [2, \ldots, 2], [3, 3, 3, 3], [7, 2, 1, 1, 1]; 2)$	$(2, 3, 7)$	Klein
$(12; [2, \ldots, 2], [3, 3, 3, 3], [8, 1, 1, 1, 1]; 2)$	$(2, 3, 8)$	$\sqrt{t}$ or Octahedral
$(18; [2, \ldots, 2], [3, \ldots, 3], [7, 7, 1, 1, 1, 1]; 2)$	$(2, 3, 7)$	$\sqrt{t}$

Here $\square$ represents any of the possible entries as listed in Theorem 4.4.

Note that in the case of the arithmetic triangle group $\mathrm{PSL}(2, \mathbb{Z})$, with triangle $(2, 3, \infty)$, as expected we recover from this list the topological types of the Kodaira functional inveriants of our five families. In this corollary, the restriction to arithmetic Fuchsian triangle groups is for convenience only — we just wanted a finite set of triangle groups in $\mathrm{PSL}(2, \mathbb{R})$ to which to apply our criterion, and in this case they yielded a finite list of topological types. By contrast, for some triangles one can explicitly construct infinite lists of allowable topological types (unlike the previous result, the proofs of these corollaries do not produce an exhaustive list of types, merely an infinite one):

Corollary 4.7. *There are infinitely many topological types corresponding to algebraic Painlevé VI solutions arising by pullback from each triangle uniformized by $\mathbb{C}$, except for $(3, 3, 3)$ which has none.*



$\mathbb{P}^1$
2
1
8
1
2
2
2
2
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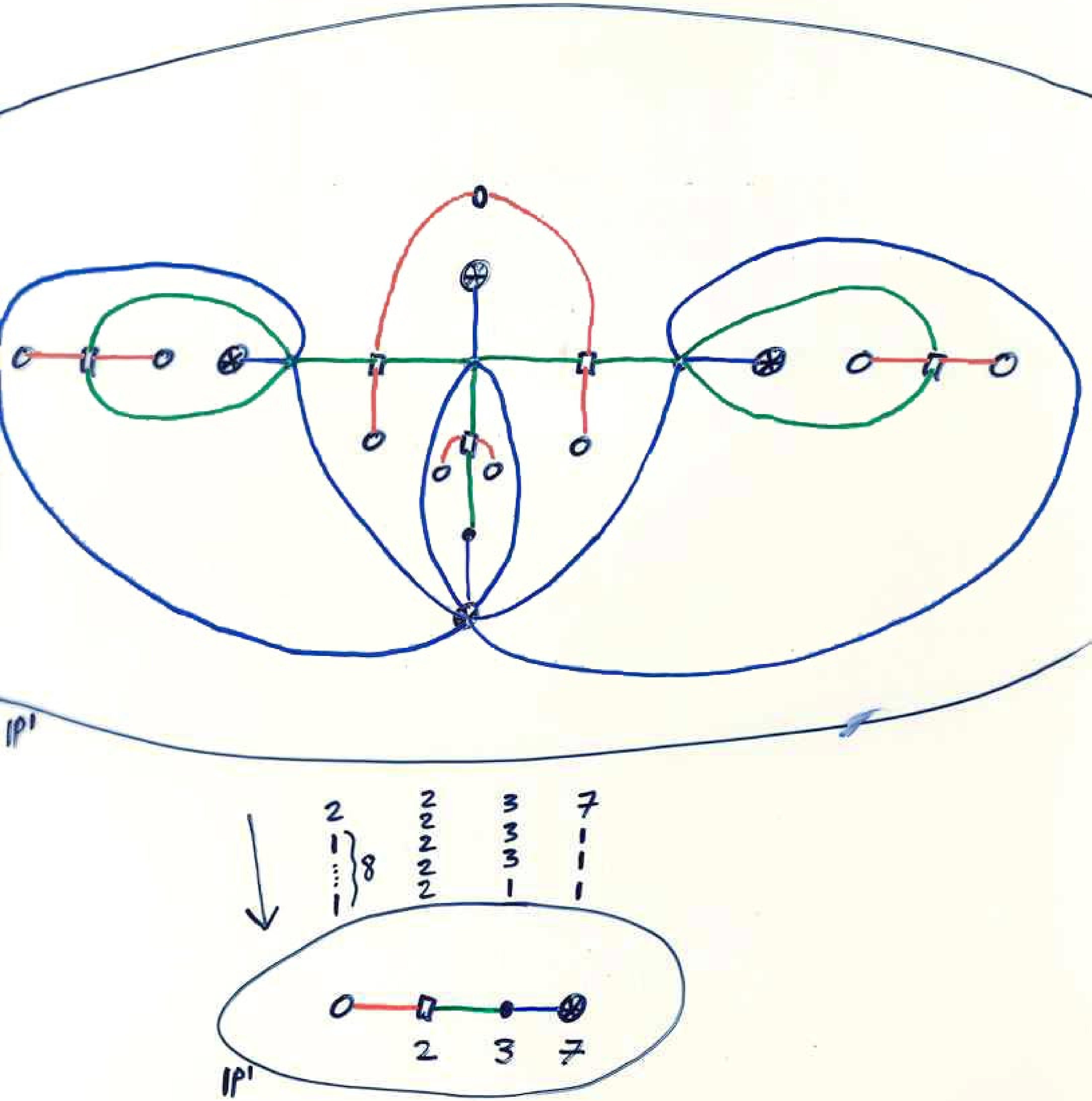
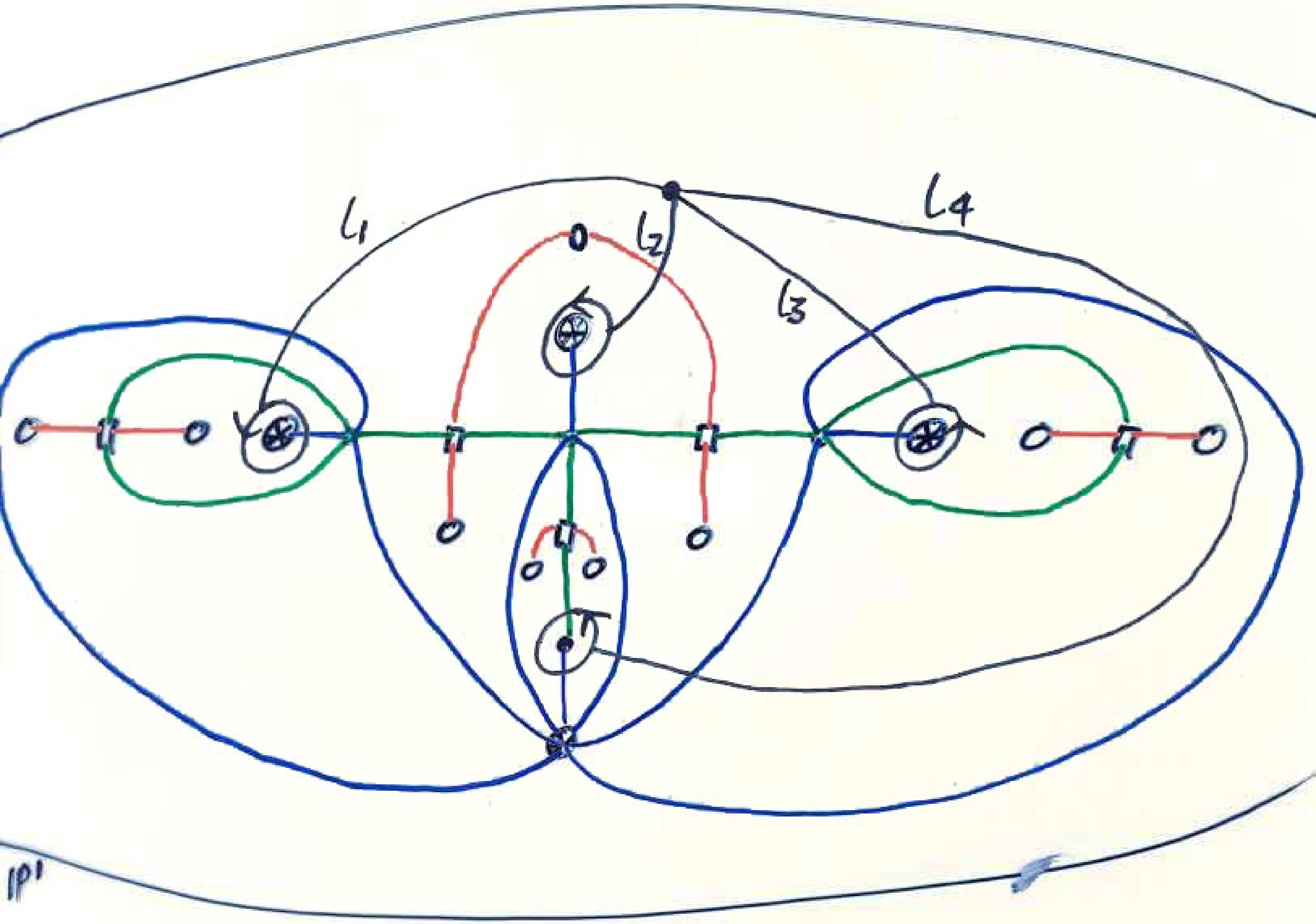
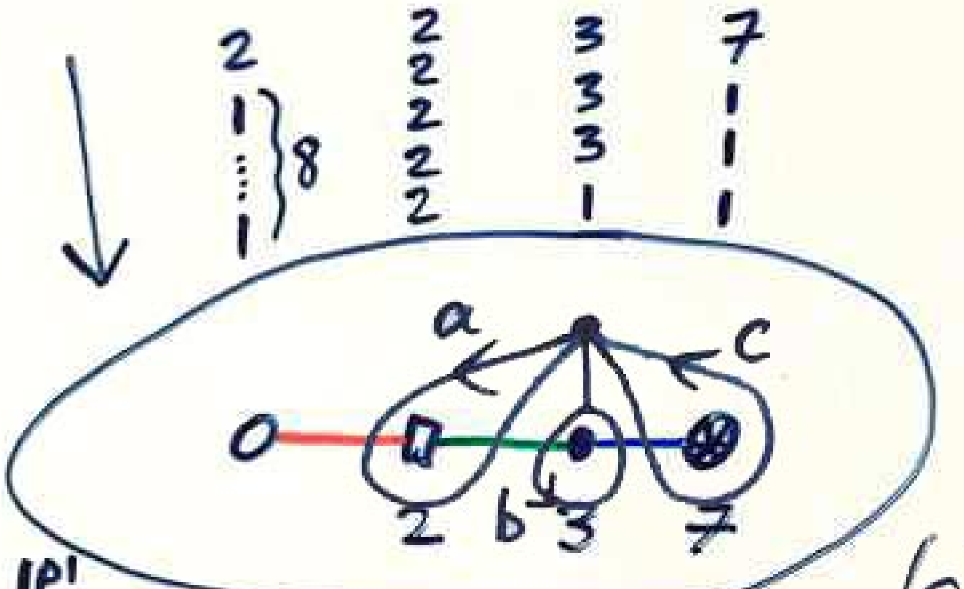
3
3
3
1
7
1
1
1
2
3
7
$\mathbb{P}^1$



P¹
2 1 ⋯ 1 } 8
2 2 2 2 2
3 3 3 1
7 1 1 1
2 3 7
P¹





$$
\begin{aligned}
l_1 &= c a c a^{-1} c^{-1} \\
l_2 &= c \\
l_3 &= c^{-1} a^{-1} c a c \\
l_4 &= c^{-3} b c^{3}
\end{aligned}
$$

$(cba=1)$

Simple observation

Can write down topological P_{VI} solution
from topology of f, by hand
(don't need f explicitly)

- go through Doran's list & find top. solutions
- compute explicitly by previous asymptotic method.

2, 3, 7 solution

genus one, 18 branches

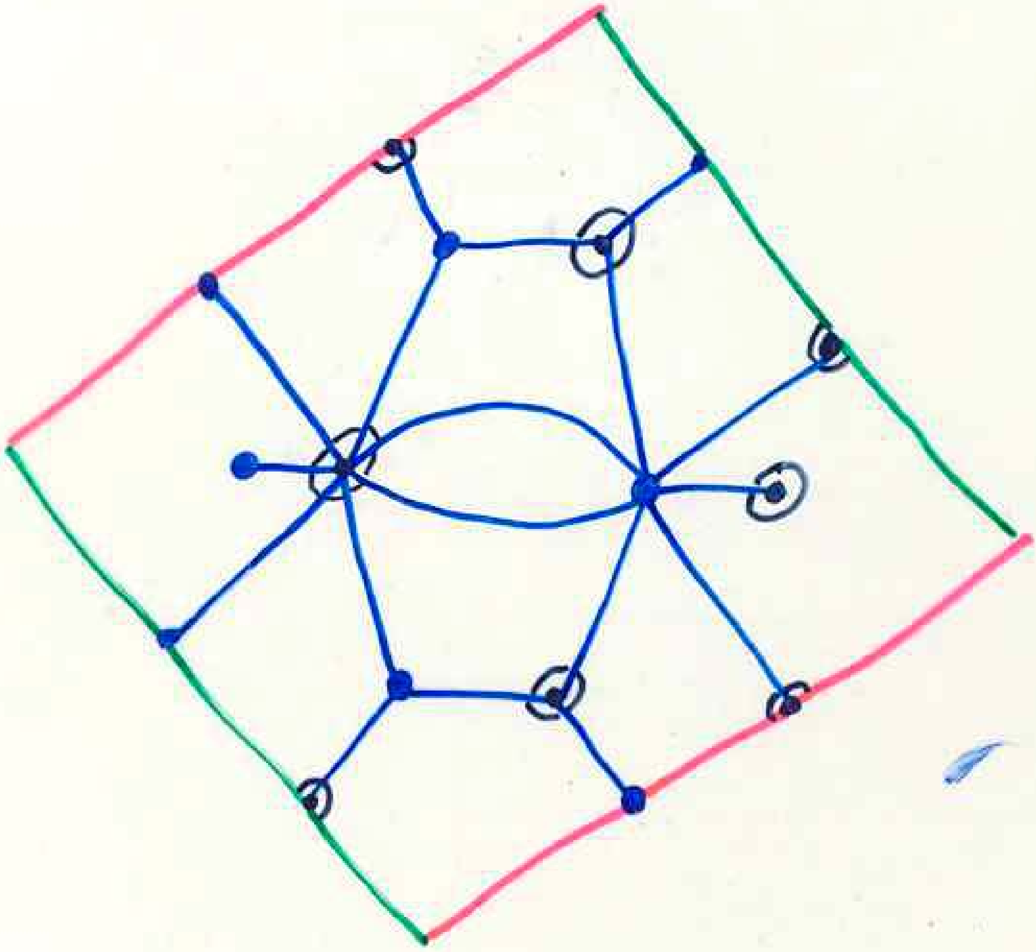
$(\theta_1, \theta_2, \theta_3, \theta_4) = (2/7, 2/7, 2/7, 1/3)$

$$y = \frac{1}{2} - \frac{\left(3\,s^8 - 2\,s^7 - 4\,s^6 - 204\,s^5 - 536\,s^4 - 1738\,s^3 - 5064\,s^2 - 4808\,s - 3199\right)u}{4\,\left(s^6 + 196\,s^3 + 189\,s^2 + 756\,s + 154\right)\left(s^2 + s + 7\right)\left(s + 1\right)}$$

$$t = \frac{1}{2} - \frac{\left(s^9 - 84\,s^6 - 378\,s^5 - 1512\,s^4 - 5208\,s^3 - 7236\,s^2 - 8127\,s - 784\right)u}{432\,s\,\left(s + 1\right)^2\left(s^2 + s + 7\right)^2}$$

where

$$u^2 = s\,(s^2 + s + 7).$$



— thanks to M. van Hoeij

Icosahedral solution 41

genus one, 18 branches

$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/3, 1/3, 1/3)$

$$y = \frac{1}{2} - \frac{8\,s^7 - 28\,s^6 + 75\,s^5 + 31\,s^4 - 269\,s^3 + 318\,s^2 - 166\,s + 56}{18\,u\,(s-1)\,(3\,s^3 - 4\,s^2 + 4\,s + 2)}$$

$$t = \frac{1}{2} + \frac{(s+1)\left(32\,(s^8+1) - 320\,(s^7+s) + 1112\,(s^6+s^2) - 2420\,(s^5+s^3) + 3167\,s^4\right)}{54\,u^3\,s\,(s-1)}$$

where $\qquad u^2 = s\,(8\,s^2 - 11\,s + 8).$

(Equivalent to Dubrovin–Mazzocco's 10 page elliptic solution.)

The corresponding family of connections on $\mathbb{P}^1$ with icosahedral monodromy is:

$$d - \left(\frac{B_1}{z} + \frac{B_2}{z-t} + \frac{B_3}{z-1}\right) dz, \quad \text{where}$$

$$B_1 = \begin{pmatrix} \lambda_1 & b_{12} & b_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ b_{21} & \lambda_2 & b_{23} \\ 0 & 0 & 0 \end{pmatrix}, \qquad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & \lambda_3 \end{pmatrix}$$

$$b_{12} = \lambda_1 \; - \mu_3 y + (\mu_1 - xy)(y-1), \qquad b_{32} = (\mu_2 - \lambda_2 - b_{12})/t,$$

$$b_{13} = \lambda_1 t - \mu_3 y + (\mu_1 - xy)(y-t), \qquad b_{23} = (\mu_2 - \lambda_3)t - b_{13},$$

$$b_{21} = \lambda_2 + \frac{\mu_3(y-t) - \mu_1(y-1) + x(y-t)(y-1)}{t-1}, \qquad b_{31} = (\mu_2 - \lambda_1 - b_{21})/t$$

$$\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}, \qquad \mu_1, \mu_2, \mu_3 = \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$$

$$x = \frac{24\,(s-1)\,(3\,s^3 - 4\,s^2 + 4\,s + 2)\,P(s)\,u}{5\,(6\,s^2 - 2\,s + 1)\,(4\,s^4 + 4\,s^3 + 54\,s^2 - 86\,s + 49)\,(2\,s-1)^2\,(2\,s^2 + s + 2)^2\,(s-2)^4}$$

$$P = 114\,s^9 - 416\,s^8 + 1184\,s^7 + 814\,s^6 - 6016\,s^5 + 9136\,s^4 - 6634\,s^3 + 2716\,s^2 - 364\,s + 91.$$

24 branch Valentiner solution

(Icosahedral Solution 46)

$$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/3, 1/3, 1/3, 1/2)$$

$$y = \frac{1}{2} - \frac{\mathrm{P}}{2\,(3\,s^2 - 2\,s + 2)\,\mathrm{R}\,u}, \quad t = \frac{1}{2} + \frac{(s^2 + 4\,s - 2)\,\mathrm{Q}}{2\,(s+2)\,(3\,s^2 - 2\,s + 2)^2\,u^3}$$

where

$$\mathrm{P} = 16\,s^{11} + 72\,s^{10} + 50\,s^9 - 242\,s^8 - 3143\,s^7 + 6562\,s^6 - 8312\,s^5 + 9760\,s^4 - 9836\,s^3 + 6216\,s^2 - 2288\,s + 416,$$
$$\mathrm{Q} = 8\,s^{10} + 16\,s^9 + 24\,s^8 - 84\,s^7 + 429\,s^6 - 312\,s^5 + 258\,s^4 - 288\,s^3 + 288\,s^2 - 128\,s + 32,$$
$$\mathrm{R} = 26\,s^6 + 18\,s^5 - 75\,s^4 + 50\,s^3 + 270\,s^2 - 312\,s + 104,$$

and where (u, s) lies on the elliptic curve

$$u^2 = (8\,s^2 - 7\,s + 2)\,(s+2).$$

Icosahedral solutions with $\geqslant 5$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)
20	5	0	1	b^2cd	44, 12, 12, 4	480	S_5
21	5	0	2	c^2d^2	36, 12, 0, 0	240	S_5
22	6	0	1	bc^2d	34, 10, 2, 2	576	S_6
23	6	0	1	bcd^2	46, 14, 10, 2	576	S_6
24	8	0	1	ac^2d	39, 15, 3, 3	768	A_8
25	8	0	1	acd^2	45, 9, 9, 3	768	A_8
26	9	1	2	bc^3	28, 4, 4, 4	288	A_9
27	9	1	2	bd^3	52, 8, 8, 4	288	A_9
28	10	0	2	a^2cd	48, 12, 6, 6	480	$2^7\,3\,5$
29	10	0	2	b^3c	46, 14, 14, 6	320	A_{10}
30	10	0	2	b^3d	42, 2, 2, 2	320	A_{10}
31	10	0	3	c^4	24, 0, 0, 0	80	A_{10}
32	10	0	3	d^4	48, 0, 0, 0	80	A_{10}
33	12	0	0	$abcd$	43, 11, 7, 1	2304	A_{12}
34	12	1	1	abc^2	37, 13, 5, 5	1152	A_{12}
35	12	1	1	abd^2	49, 5, 5, 1	1152	A_{12}
36	12	1	1	b^2cd	38, 6, 6, 2	1152	$2^9\,3^2\,5$
37	15	1	2	b^3c	36, 4, 4, 4	480	A_{15}
38	15	1	2	b^3d	48, 8, 8, 8	480	A_{15}
39	15	1	2	b^2c^2	32, 8, 0, 0	720	S_{15}
40	15	1	2	b^2d^2	44, 4, 0, 0	720	S_{15}
41	18	1	3	b^4	40, 0, 0, 0	144	$2^{14}\,3^4\,5\,7$
42	20	1	1	ab^2c	41, 9, 9, 1	1920	A_{20}
43	20	1	1	ab^2d	47, 7, 3, 3	1920	A_{20}
44	20	1	3	a^2c^2	42, 18, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
45	20	1	3	a^2d^2	54, 6, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
46	24	1	2	ab^3	45, 5, 5, 5	768	$2^{20}\,3^5\,5^2\,7\,11$
47	30	2	2	a^2bc	46, 14, 4, 4	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
48	30	2	2	a^2bd	52, 8, 2, 2	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
49	36	3	3	a^2b^2	50, 10, 0, 0	864	$2^{23}\,3^4\,5\,7$
50	40	3	3	a^3c	51, 9, 9, 9	320	$2^{25}\,3^4\,5^2\,7$
51	40	3	3	a^3d	57, 3, 3, 3	320	$2^{25}\,3^4\,5^2\,7$
52	72	7	3	a^3b	55, 5, 5, 5	576	$2^{32}\,3^4\,5\,7$

Quadratic / Landen / Folding transformations

Kitaev Manin Tsuda–Okamoto–Sakai

Kitaev's perspective:

If A a fuchsian system with poles at $0, t, 1, \infty$

& with (proj.) monodromy of order 2 at $0, \infty$

- pullback A along $z \mapsto z^2$
- get system B with 4 non-apparent sing.s at $\pm 1, \pm\sqrt{t}$
- remove apparent sings & renormalize

(*) IMDs of $A \Leftrightarrow$ IMDs of resulting system (*)

$\leadsto$ get transform relating certain P_{VI} solutions (codim 2 in param. space)

- Much simpler explicit formulae for transform later (conjugate by Okamoto transformations) (Ramani, Grammaticos, Tamizhmani 2000)

Theorem (Ramani–Grammaticos–Tamizhmani 2000)
Given a solution (y_0, t_0) of P_{VI} with parameters of the form

$$\theta = (0, \theta_2, \theta_3, 1)$$

then, by taking two square roots, one obtains a new solution (y, t) with parameters

$$\theta = (\theta_3, \theta_2, \theta_2, 2 - \theta_3)/2$$

where

$$y = \frac{(\tau - 1)(\eta + 1)}{(\tau + 1)(\eta - 1)}, \qquad t = \left(\frac{\tau - 1}{\tau + 1}\right)^2$$

with

$$\eta^2 = y_0, \qquad \tau^2 = t_0.$$

Theorem′ (Tsuda–Okamoto–Sakai 2005)
Given a solution (y_0, t_0) of P_{VI} with parameters of the form

$$\theta = (\theta_1, \theta_2, \theta_2, 1 - \theta_1)$$

then, by taking one square root, one obtains a new solution (y, t) with parameters

$$\theta = (0, 2\theta_2, 0, 1 - 2\theta_1)$$

where

$$y = \frac{1}{2} + \frac{1}{4}\left(\frac{\tau}{y_0} + \frac{y_0}{\tau}\right)$$
$$t = \frac{1}{2} + \frac{1}{4}\left(\tau + \frac{1}{\tau}\right)$$

with

$$\tau^2 = t_0.$$

Corollary ("Unfolding transformation")

If functions y_0, t_0 of the form

$$y_0 = \frac{1}{2} + a_y(s)u, \qquad t_0 = \frac{1}{2} + a_t(s)u$$

are a $\mathrm{P_{VI}}$ solution with parameters

$$\theta = (0, \theta_2, 0, \theta_4)$$

on a Painlevé curve of the form

$$\Pi := \{u^2 = u_2(s)\}$$

for a polynomial $u_2(s)$, then the functions

$$y = \frac{1}{2} + \frac{w+v}{2(A_y - A_t)}, \qquad t = \frac{1}{2} - \frac{A_t}{2w}$$

are a $\mathrm{P_{VI}}$ solution for parameters

$$\theta = (1 - \theta_4, \theta_2, 1 - \theta_4, 2 - \theta_2)/2$$

on the curve obtained by adjoining to $\mathbb{C}(s)$ the functions v, w where

$$v^2 = A_y^2 - u_2, \qquad w^2 = A_t^2 - u_2$$

and $A_i = 2a_iu_2$ for $i = y, t$.

Icosahedral solutions with $\geqslant 5$ branches

	Degree	Genus	Walls	A_5 Type	Alcove Point	No.	Group (size)
20	5	0	1	$b^2\,c\,d$	44, 12, 12, 4	480	S_5
21	5	0	2	$c^2\,d^2$	36, 12, 0, 0	240	S_5
22	6	0	1	$b\,c^2\,d$	34, 10, 2, 2	576	S_6
23	6	0	1	$b\,c\,d^2$	46, 14, 10, 2	576	S_6
24	8	0	1	$a\,c^2\,d$	39, 15, 3, 3	768	A_8
25	8	0	1	$a\,c\,d^2$	45, 9, 9, 3	768	A_8
26	9	1	2	$b\,c^3$	28, 4, 4, 4	288	A_9
27	9	1	2	$b\,d^3$	52, 8, 8, 4	288	A_9
28	10	0	2	$a^2\,c\,d$	48, 12, 6, 6	480	$2^7\,3\,5$
29	10	0	2	$b^3\,c$	46, 14, 14, 6	320	A_{10}
30	10	0	2	$b^3\,d$	42, 2, 2, 2	320	A_{10}
31	10	0	3	c^4	24, 0, 0, 0	80	A_{10}
32	10	0	3	d^4	48, 0, 0, 0	80	A_{10}
33	12	0	0	$a\,b\,c\,d$	43, 11, 7, 1	2304	A_{12}
34	12	1	1	$a\,b\,c^2$	37, 13, 5, 5	1152	A_{12}
35	12	1	1	$a\,b\,d^2$	49, 5, 5, 1	1152	A_{12}
36	12	1	1	$b^2\,c\,d$	38, 6, 6, 2	1152	$2^9\,3^2\,5$
37	15	1	2	$b^3\,c$	36, 4, 4, 4	480	A_{15}
38	15	1	2	$b^3\,d$	48, 8, 8, 8	480	A_{15}
39	15	1	2	$b^2\,c^2$	32, 8, 0, 0	720	S_{15}
40	15	1	2	$b^2\,d^2$	44, 4, 0, 0	720	S_{15}
41	18	1	3	b^4	40, 0, 0, 0	144	$2^{14}\,3^4\,5\,7$
42	20	1	1	$a\,b^2\,c$	41, 9, 9, 1	1920	A_{20}
43	20	1	1	$a\,b^2\,d$	47, 7, 3, 3	1920	A_{20}
44	20	1	3	$a^2\,c^2$	42, 18, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
45	20	1	3	$a^2\,d^2$	54, 6, 0, 0	480	$2^{17}\,3^4\,5^2\,7$
46	24	1	2	$a\,b^3$	45, 5, 5, 5	768	$2^{20}\,3^5\,5^2\,7\,11$
47	30	2	2	$a^2\,b\,c$	46, 14, 4, 4	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
48	30	2	2	$a^2\,b\,d$	52, 8, 2, 2	1440	$2^{24}\,3^6\,5^3\,7^2\,11\,13$
49	36	3	3	$a^2\,b^2$	50, 10, 0, 0	864	$2^{23}\,3^4\,5\,7$
50	40	3	3	$a^3\,c$	51, 9, 9, 9	320	$2^{25}\,3^4\,5^2\,7$
51	40	3	3	$a^3\,d$	57, 3, 3, 3	320	$2^{25}\,3^4\,5^2\,7$
52	72	7	3	$a^3\,b$	55, 5, 5, 5	576	$2^{32}\,3^4\,5\,7$

Solution 52

72 branches, genus 7

$(\theta_1, \theta_2, \theta_3, \theta_4) = (1/12, 1/12, 1/12, 11/12)$

$$y = \frac{1}{2} + \frac{9\,(j-1)\,(j^3+27j^2-57j+79)\,wv + 2\,(2j^2-2j+5)\,(j^2-7j+1)\,(2j^4+2j^3-3j^2-58j+107)\,(j^2-4j+13)^2}{6\,(j^2-1)\,(2j^2+j+17)\,(j^3-3j^2+3j-11)\,(2j-7)^2\,v}$$

$$t = \frac{1}{2} + \frac{(s+1)\left(32\,(s^8+1) - 320\,(s^7+s) + 1112\,(s^6+s^2) - 2420\,(s^5+s^3) + 3167\,s^4\right)}{54\,s\,(s-1)\,u^3}$$

on the curve in $\mathbb{P}^3$ with affine equations:

$$v^2 = -\,(j+1)\left(6+j^2-2\,j\right)\left(4\,j^2-13\,j+19\right),$$
$$w^2 = (j-1)\,(2\,j-7)\,(j+1)\left(2\,j^2+j+17\right)\left(4\,j^2-13\,j+19\right)$$

where

$$s = \frac{j^2-1}{2j-7}, \qquad u = \frac{w}{(2j-7)^2}.$$

In fact this genus 7 curve is birational to the plane octic cut out by:

$$9\,(p^6\,q^2+p^2\,q^6)+18\,p^4\,q^4+4\,(p^6+q^6)+26\,(p^4\,q^2+p^2\,q^4)+8\,(p^4+q^4)+57\,p^2\,q^2+20\,(p^2+q^2)+16$$

Problems

- Prove there are no more algebraic solutions

- Is there another embedding of P_{VI} in the Schlesinger system s.t. the $g=1$ 237 solutions controls IMDs of fuchsian systems with finite monodromy?

- Why are the Painlevé curves Π defined $/\mathbb{Q}$?

- Extend Hitchin's twistor viewpoint to the icosahedral solutions
  ~ rational curves in Umemura-Mukai's 3-fold